Graph Theory Introduction!

A **graph** G consists of two sets: a set of **vertices** V and a set of **edges** E. A vertex is simply a labeled point. An edge is a connection between two vertices. For example, suppose we have vertex set $V = \{a, b, c, d\}$ and edge set $E = \{(a, b), (a, c), (a, d), (c, d)\}$. The coordinate notation (a, b) means an edge connecting vertex a and vertex b. In this way, we can see a visual representation of the corresponding graph G with 4 vertices and 4 edges below:

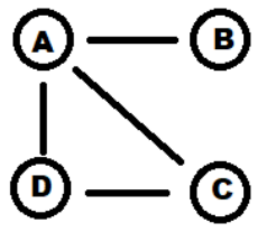


Figure 1: Our first Graph G with vertices V and edges E as above

Note that the points depict the vertices and the lines depict the edges. The graph above is quite simple. Indeed, we can draw much more complicated graphs: such as graphs with multiple edges (meaning that a pair of vertices can be connected by several edges) and loops (meaning that a vertex can be connected to itself). A **simple graph** is a graph without any multiple edges or loops. In this handout, you can assume that all the graphs we deal with are simple graphs.

Graphs are used to mathematically represent all sorts of different real-life situations! In particular, they typically are used to represent some sort of connection between interesting objects. Consider the following examples.

Example 1. *We can draw a friendship graph for the Intermediate 1 students! The vertices of this graph would represent the students in Intermediate 1. Then, there is an edge between two students if and only if the two students are friends with each other.*

Example 2. *Suppose you are looking at a map of countries. You can then draw a graph where the vertices represent the countries. Then, you can draw an edge between two vertices if the corresponding countries are neighbors, meaning they share a common border.*

If we can mathematically understand graphs, we can better understand a lot of real-words objects, phenomena, and their connections. Because of this, **graph theory** is essential in fields such as computer science and economics!

Problem 1. *Given the map of South American countries below, draw the corresponding graph described by Example 2.*

Problem 2. *Seven schoolchildren participated in a chess tournament. It is known that Misha played 6 games, Kolya - 5, Ilya and Grisha - three each, Andrey and Seva - two each, and Maxim - one. No two kids played each other twice. Who did Ilya play with?*

Problem 3. *Write the numbers* $0, 1, 2, \ldots, 15$ *in a line so that the sum of any two adjacent numbers is a square number. For example,* 6 *and* 10 *can be next to each other since they add up to* 16*, which equals* 4^2 *(but that doesn't mean* 6 *and* 10 *have to be next to each other).*

Degree of a vertex

Recall, each edge is associated to two vertices. For edge $e = (a, c)$, e is **incident** to vertices a and c. Vertices a and c are **adjacent**, as there is a common edge incident to both of them. For a vertex v in the vertex set V, we define the **degree** of v, written $deg(v)$, to be the total number of edges incident to vertex v. In our graph of South American countries, the degree of a country would correspond to the number of other countries it shares a border with.

Problem 4. *Calculate the degree of each vertex in the graph G from Figure 1:*

(i) $deg(A) =$

(ii) $deg(B) =$

(iii) $deg(C) =$

(iv) $deg(D) =$

A vertex v is **even** if $deg(v)$ is even. Similarly, v is **odd** if $deg(v)$ is odd.

Problem 5. *Answer the following questions about graph G from Figure 1:*

(i) *How many even vertices does G have?*

(ii) *How many odd vertices does G have?*

Problem 6. *Draw your own graph below, with at least 5 vertices and 5 edges. We will refer to this graph as G', with vertex set V' and edge set E'.*

(i) *How many even vertices does G' have?*

(ii) *How many odd vertices does G' have?*

(iii) *Examine your neighbors' graphs and their answers to* ***(ii)***. *Make a hypothesis about the number of odd vertices in an arbitrary graph.*

(iv) *Draw more graphs below to either affirm or contradict your hypothesis. Continue until you are confident in your answer.*

After playing with a few more graphs, you should notice that the number of odd vertices seems to always be even. This turns out to be true and is famous enough to have a name: the **Handshaking lemma**. Intuitively, the lemma says in a party of people, some of whom shake hands with each other, an *even* number of people must shake an odd number of hands. This fact was first proved by Euler many centuries ago, let's see how!

Problem 7. *Suppose we have a graph G with vertex set V and edge set E, consisting of n vertices and m edges.*

(i) *Let e be an edge in E. How many vertices is e incident to? (Note: each point where an edge meets a vertex contributes 1 to the degree of that vertex.)*

(ii) *Using your answer to* ***(i)****, argue that if we add together the degree of every vertex in V, the result will be even. (Bonus: what will the sum be exactly?)*

(iii) *If we add together the degree of every even vertex, will the result be even or odd?*

(iv) *Use* ***(ii)*** *and* ***(iii)*** *to argue that if we add together the degree of every odd vertex, the result must be even.*

(v) *Use* ***(iv)*** *to argue that the total number of odd vertices is even.*

Complete graphs

A **complete graph on n vertices** is a special type of simple graph, which we denote by K_n. It has n vertices, and each pair of distinct vertices has an edge connecting them. In other words, every vertex is connected to every other vertex.

Problem 8. *Draw K_2, K_3, K_4, and K_5.*

Problem 9. *How many edges are there in a complete graph with n vertices?*

Problem 10. *Complete the following parts.*

(i) *What is the degree of each vertex in K_3? Is the degree of each vertex the same or different?*

(ii) *What is the degree of each vertex in K_4? Is the degree of each vertex the same or different?*

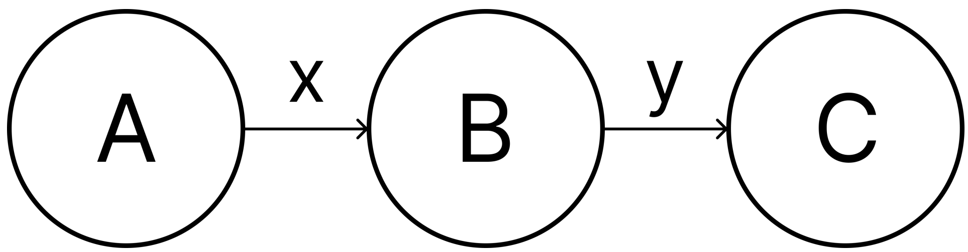
(iii) *What is the degree of each vertex in K_n for any integer $n \geq 1$? You should write your answer as an equation in terms of n.*

(iv) *For what values of n will each vertex in K_n have even degree? Why these values of n?*

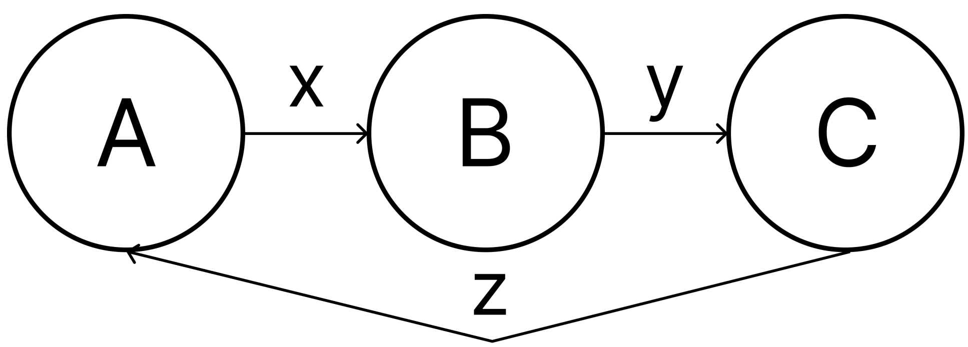
(v) *For what values of n will each vertex in K_n have odd degree? Why these values of n?*

Paths and connected graphs

A **path of length** n along a Graph G is a sequence of distinct vertices and edges connecting adjacent vertices: V_1, e_1, V_2, e_2, $\ldots$, e_n, V_{n+1}. So, e_1 is (v_1, v_2), e_2 is (v_2, v_3), and so on until e_n which is (v_n, v_{n+1}). Note that we do not allow vertices or edges to repeat in a path. The **length** of the path is simply the number of edges along the path. For example, the following path of length 2 can be written as: $AxByC$.



A **cycle** is a special kind of path where the only vertex that repeats is the starting and ending vertex. This means that $V_1 = V_{n+1}$ while e_1, $\ldots$, e_n are all still distinct (and the vertices $V_2, \ldots, V_n$ are also distinct and all not equal to V_1). We can change the above path into a cycle like so:



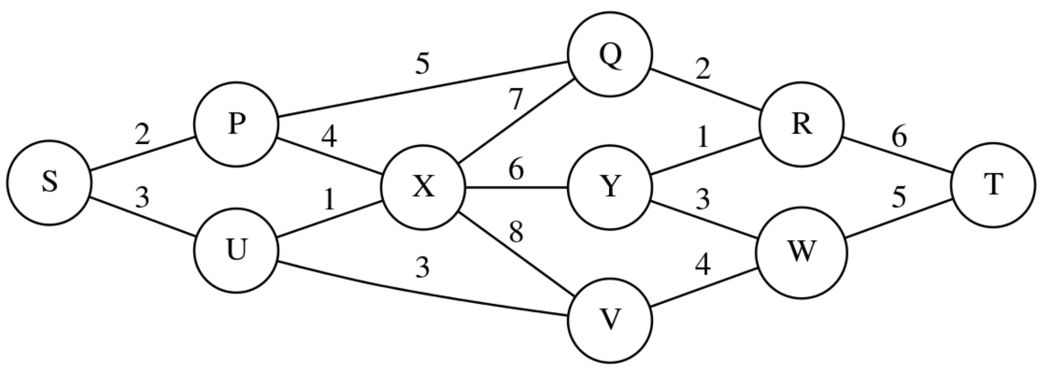
The cycle above is written as: $AxByCzA$

Problem 11. *Recall the graph from Figure 1, with vertex set* $V = \{a, b, c, d\}$ *and edge set* $E = \{(a, b), (a, c), (a, d), (c, d)\}$. *Complete the following parts.*

(i) *How many paths from a to d are there? Write them down in the path notation above.*

(ii) *How may cycles are there? Write them down in the path notation above.*

Problem 12. *Consider the following graph:*

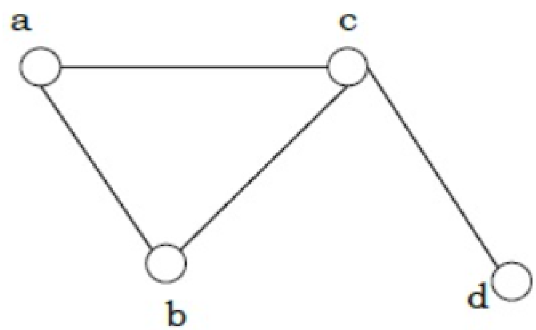


The numbers on the edges represent the cost of traveling from one vertex to another. For example, it costs 2 sand dollars to go from S *to* P, *and 3 sand dollars to go from* S *to* U. *Can you find the cheapest path from* S *to* T? *Compare your answer with a neighbor.*

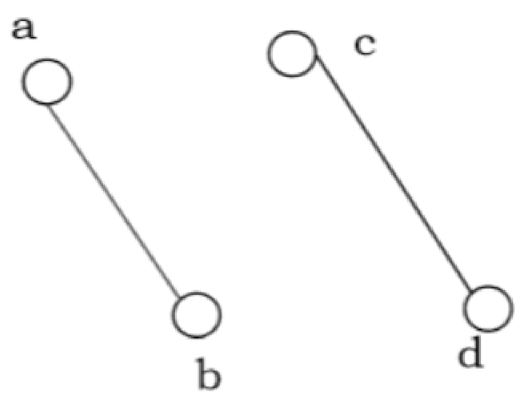
A graph is **connected** if there is a path connecting any two vertices. It is easy to visually tell if a graph is connected or not.

Problem 13. *Complete the following parts.*

(i) *Is the graph below connected?*



(ii) *Is the graph below connected?*



Problem 14. *Suppose that a graph G is just a cycle. Prove that G must be connected.*

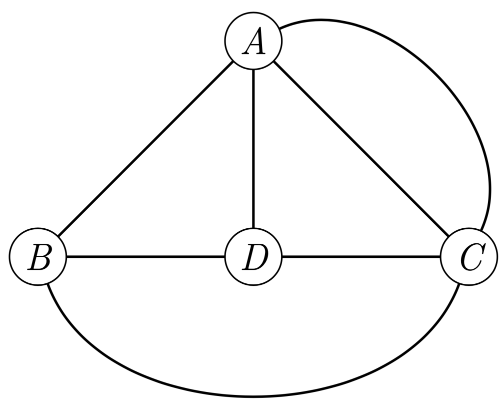
Problem 15. *Suppose that every vertex in a graph G is of degree 2. Prove that G must be a cycle. So, is G connected?*

Eulerian paths and cycles

An **Eulerian path** is a path that traverses each edge exactly once. An **Eulerian cycle** is an Eulerian path that starts and ends at the same vertex. They are named in honor of a legendary Swiss mathematician, Leonhard Euler (1707-1783), considered by many as the founding father of graph theory.

Problem 16. *Can there be repeated edges in an Eulerian cycle? What about an Eulerian path? Why or why not?*

Problem 17. *Consider the following graph:*



(i) *Does the following graph have an Eulerian path? Why or why not?*

(ii) *Does the graph have an Eulerian cycle? Why or why not?*

Problem 18. *Recall that K_n is a complete graph with n vertices.*

(i) *Draw the graph K_5 below. Does it have an Eulerian cycle?*

(ii) *Does the graph K_6 have an Eulerian cycle?*

(iii) *What about K_7?*

(iv) *Do you notice a pattern? Can you (without proof) give a condition that a graph must satisfy to contain an Eulerian cycle?*