Logarithms and Big Numbers

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Quick Warm Up: Guess for which integer k is 10^k closest to 52!? We just would like to see how small/large this numbers seems to you.

Exercise 1. *Before we move on, let us do a brief review of laws of exponents. For all of the following problems you don't have to compute the exact decimal number. You just have to simplify the algebraic expression.*

(a) Can you simplify $2^{13}2^7$*? No need to compute the exact value, just simplify it into one expression.*

(b) Can you simplify $3^8 7^8$*?*

(c) Can you simplify $2^{13}8^{-4}$*?*

(d) Please simplify $2^{(3^4)}$ *and* $(2^3)^4$*. Note that these are not the same thing.*

A very useful function that you might not have heard of before is called the logarithm (usually shortened to just log). The log is defined as follows:

$\log_a(x) = b$ is the same thing as $a^b = x$ as long as $a > 0$.

So for example:

$\log_2(8) = 3$ because $2^3 = 8$

$\log_{10}(100) = 2$ because $10^2 = 100$

$\log_9(3) = 1/2$ because $9^{1/2} = 3$

$\log_4(1/16) = -2$ because $4^{-2} = 1/16$

The expression $\log_a(x) = b$ is read "The logarithm of x in the base a equals b." The logarithm might look strange but trust me, it is very useful!

Exercise 2. *(a) What's the value of* $\log_6(36)$

(b) What's the value of $\log_5(125)$

(c) What's the value of $\log_{81}(3)$

(d) What's the value of $\log_{12395817}(12395817)$*? This one looks hard, but is actually quite easy!*

(e) Suppose that $\log_{131}(x) = 0$*. What is x?*

(f) What is $\log_7(765)$*? This question is either very easy or very difficult. Check with your instructor if you get stuck.*

(g) What are the values of $\log_5(1/5)$, $\log_5(0)$, *and* $\log_5(-5)$*?*

Logarithms and exponents are very closely related (just like addition/subtraction and multiplication/division). For every law of exponents there is an equivalent law of logarithms. For example think about the law $a^b a^c = a^{b+c}$. If we notate a^b as x and a^c as y, then we get:

$a^b = x$ and $a^c = y$ so multiplying the equations together gives $a^b a^c = xy$ but also $a^b a^c = a^{b+c}$ so $a^{b+c} = xy$.

By converting the above into logarithms we get that:

$\log_a(x) = b$

$\log_a(y) = c$

$\log_a(xy) = b + c$

Therefore, we have that: $\log_a(x) + \log_a(y) = \log_a(xy)$

and voila! We proved one of the laws of logarithms! Notice that our final answer is just a statement about logarithms, and exponents are nowhere to be seen.

Exercise 3. *(a) Find the logarithm law that corresponds to the law of exponents, $(a^b)^c = a^{bc}$. Have an instructor check your answer. *Hint, let $x = ab$.*

(b) Prove the change of base formula for logarithms, that is that $\frac{\log_a(x)}{\log_a(b)} = \log_b(x)$.

(c) Using the above laws, prove that exponentiation base a and logarithms base a are inverses of each other. In other words, prove that $a^{\log_a(x)} = x$ and also that $log_a(a^x) = x$.

Exercise 4. *Show that* $\log_a(x^y) = y \log_a(x)$ *for any* x*,* y *such that* $x^y > 0$*.*

Exercise 5. *What does* $\log_3(2187) - log_3(9^2)$ *equal? Use the rules you proved from above.*

A common convention in some scientific fields is that a log without a base is $\log_{10}$. From now on, if you see a logarithm without a base, you can assume that the base is base 10. Note that this is not a common mathematical convention however. Please solve the following log questions!

Exercise 6. *(a) What are the values of the following?*

$\log(10) =$

$\log(100) =$

$\log(1000) =$

$\log(1000000) =$

$\log(0.0001) =$

(b) Mathematicians say that the logarithm is strictly increasing. This is exactly what you think it means, it just means that the larger number you put into it, the larger number you get out. In mathematical terms, if $a > b$ *then* $\log(a) > \log(b)$ *as well. Show this fact.*

(c) The company Google is named after the number googol, which is defined as 1 with one hundred zeros after it. What is $\log(googol)$*? A googolplex* $= 10^{googol}$*. What is* $\log(googolplex)$*?*

Approximate log chart:

x	$\log(x)$
1	0
2	0.30
3	0.48
4	0.60
5	0.70
6	0.78
7	0.85

8	0.90
9	0.95
10	1
11	1.04
12	1.08
13	1.11
14	1.15
15	1.18

Exercise 7. *Just how large is* 52! *really?*

(a) As a little warm-up, please compute 1!, 2!, 3!, 4! and 5!.

(b) Please rate the following numbers from smallest to largest: $3 * 6$, $6!$, 3^6, 6^3

(c) Do the same for the following. Form an educated guess, but do not actually compute. $52!$, 2^{200}, 25^{52}, 52^{10}

(d) It is actually pretty difficult to estimate the relative sizes of factorials and exponents, but there is a secret. Don't compare the numbers directly. Instead compare their logarithms. Approximate $\log(2^{200})$, $\log(25^{52})$, *and* $\log(52^{10})$

(e) Please show that $\log(6!) = \log(1) + \log(2) + \log(3) + \log(4) + \log(5) + \log(6)$*. Then, compute* $\log(6!)$ *using the logarithm table.*

(f) There is a formula called Stirling's formula that approximates log(n!). The formula is:

$\log(n!) = n * \log(n) - 0.43 * n + 0.5 * \log(n) + 0.4$ *Use Stirling's formula to approximate* $\log(6!)$*. Compare this to you answer to the previous question.*

(g) Use Stirling's formula to approximate $\log(52!)$*.*

(h) How many digits before the decimal does 52! have? Use your answers from 5.d) and 5.g) to finally give

a rigorous answer to 5.c) How close was your guess?

Exercise 8. *(a) Suppose that someone spent their whole life shuffling cards so that they could see as many shuffles as possible. If they lived for 100 years, and shuffled a deck of cards once per minute how many shuffles would they have done? A year is 525,600 minutes.*

(b) Is there even a 1 percent chance that they had seen our shuffling before? Why or why not?

(c) What about there was a country of 10^9 *people who all shuffled cards all day for 100 years. Would they have a 1 percent chance of seeing our sequence of cards?*

(d) Current estimates are that there are about 100 billion people who have ever lived. If for all of human existence we lived for 100 years and did nothing other than shuffle cards, what is the likelihood that one person would have seen our sequence?

Exercise 9. *Here are some more problems, that showcase just how big* $52!$ *really is...*

(a) Let's pretend that you were looking to kill some time. Every second you take a drop of water out of the ocean and put it aside in a (very large) bucket. When all of the water on all of the oceans is in the bucket, you refill the oceans and place one grain of sand on the ground. You repeat this process and place a second grain of sand on top of the first, then you repeat it a third time, etc... until you make a pile of sand high enough to touch the moon. Would you have waited more or less than $52!$ *seconds? The ocean has about* $2.6 * 10^{25}$ *drops in it, a grain of sand is at smallest* $4 * 10^{-6}$ *meters in diameter and the moon is*

about $3.8 * 10^8$ *meters away.*

(b) As a way to kill a little more time, suppose that every time your sand pile reached the moon, you put all of the sand back and bought a power ball (the largest US lottery) ticket. Every time you won the lottery you rolled 10 dice. You did this process until you all 10 dice came up 6. How long would you have waited? More than $52!$ *seconds or not? The chances of winning powerball are about 1 in 300 million, and 610 is about* $6 * 10^7$.

Logarithms play an important role in determining how long different algorithms take to run in the field of computer science. We see some examples of this below.

Exercise 10. *Imagine you have 32 items sorted in order in a list. This could be words sorted alphabetically or maybe data points sorted in an increasing fashion. We will continue the problem assuming it is numbers in order. Say it takes 1 unit of time to "look" at a value at some index (i.e observing what element is at* *the 13th position in the list). First, develop a reasonable and fast way to find a desired element without any prior knowledge about what elements are in the list. Then figure out what the maximum amount of time this algorithm can take to find the desired element, assuming it is in fact in the list (hint: it should include* $\log_2$ *). Do you believe that your method has the smallest maximum amount of time this algorithm can take to find the desired element out of any possible algorithm? Why or why not? Work this out with a few partners and tell the instructors what you come up with.*

Exercise 11. *A complete binary tree is a data structure used in computer science. There is a root node at the top. Then it has two children. Then each of its children have two children. Continue this for however many steps you would like. One will be drawn on the board as an example. Suppose a complete binary tree has 8192 bottom nodes (often times referred to as leaf nodes). What is the height of this tree, where we define the height of the tree to be the number of genertions of leaf node can look back and see. For example,* *if a leaf node's oldest ancestor were its grandfather, it would have height 2.*

Exercise 12. *Now suppose a complete binary tree has 8191 total nodes. Now what is the height of this tree?*

After solving these exercises, we see that in computer science it is much more natural to have the base of the logarithm as 2. In fields like chemistry and other physical sciences, the logarithm with base 10 is more frequently used. In fact, in mathematics, the most important and commonly used base of a logarithm is the number e, referred to as Euler's number.