Game

Prepared by Mark on April 13, 2023

Part 1: Numbers

Problem 1:
Each letter in $MATHM + AJORS$ represents a single-digit number. Maximize this quantity.
Only the team with the highest result gets points for this problem. Ties are permitted.
You may submit as many solutions as you wish.

Problem 2:
Q is a three digit number.
$Q - 7$ is divisible by 7. $Q - 8$ is divisible by 8. $Q - 9$ is divisible by 9. What is Q?

Problem 3:
Alex and Anna share a tub of popcorn. Alex eats one kernel, Anna eats two. Alex then eats three, and the pattern continues. The person that takes the final turn consumes all the remaining popcorn, even if there aren't enough kernels for a complete turn.
Alex ate 2017 kernels. How many were left for Anna?

Problem 4:
Several natural numbers were multiplied to get 224.
The smallest of these was exactly equal to half the largest.
How many numbers were multiplied?

Problem 5:
How many natural numbers n less than 10,000 satisfy $2^n - n^2 \equiv 0 \pmod{7}$?

Part 2: Algebra

Problem 1:

$$\text{Evaluate} \ \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right) \ \ldots \ \left(1-\frac{1}{255}\right)$$

Problem 2:
$(a+b)(a+b-1) = ab$ and $a^2+b^2=3$.
Find a^3+b^3.

Problem 3:
Simplify $(2^{62}+1)/(2^{31}+2^{16}+1)$

Problem 4:
$x, y, z > 0$ and $xyz = 1$.
Also, $x + 1/z = 5$ and $y + 1/x = 29$.
Find $z + 1/y$.

Problem 5:
Factor $x^8 + x^4 + 1$ into four quadratics.

Part 3: Combinatorics

Problem 1:
A real estate investor asked a developer to paint 2017 houses so that at least 1000 are green and 1000 are red. What is the maximum number of colors the developer can use?

Problem 2:
How many ways are there to cut one 3×5 rectangle into five 1×3 rectangles?

Problem 3:
How many different integral solutions (x, y, z) are there to $x + y + z = 20$?

Problem 4:
Consider an uncolored 8×8 board. How many ways are there to paint the squares black or white so that we end up with exactly 31 black squares, none of which share an edge?

Problem 5:
Let A be the set of four-digit integers in which the first digit is equal to the sum of the other three. Let B be the set of four-digit integers in which the last digit is equal to the sum of the other three. Which set is larger, and by how many elements?

Part 4: Misc

Problem 1:
A carpenter cut a chessboard into 1×1 squares in 70 minutes.
How long will it take him to cut it into 2×2 squares?

Problem 2:
There are two kinds of books on a shelf: those on permissible magic and those on black magic. Two books on permissible magic cannot be set between exactly three other books, and two books on black magic may not stand next to each other.
What is the maximal amount of books that may be placed on the shelf?

Problem 3:
The numbers 1...9 are arranged in a 3×3 grid.
The sum of each row and column is then computed.
What is the maximum number of consecutive integers one may find in the set of these sums?

Problem 4:
16 rugby teams participate in a regional championship. Each pair of teams plays against each other twice. The 8 teams with the most wins will proceed to the national championship. If there is a tie in this ranking, the tied teams will draw lots.
Assume a rugby game can never tie. What is the minimum number of wins a team needs to guarantee a spot in the nationals?

Problem 5:
Five boxes are filled with pastries. We know that box C contains a third of the pastries in E, and that B contains two times more than C and E combined. A contains half the number of pastries in E, and a tenth of those in D. Box B contains four times more pastries than D.
What is the minimal possible positive number of pastries in all the boxes put together?

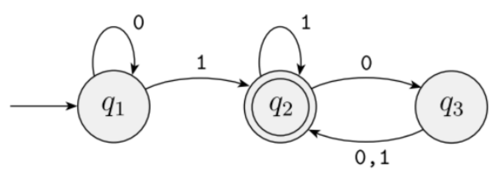
Part 5: Review

Problem 1:
Let $(G, *)$ be a finite group with identity e. Let $g \in G$.
The smallest n that satisfies $g^n = e$ defines the *order* of g.

Let a be the order of 2 in $(\mathbb{Z}_5, +)$.
Let b be the order of 4 in $(\mathbb{Z}_7^\times, \times)$.
What is ab?

Problem 2:
What is the set of the elements of $\mathbb{Z}_{14}^\times$?

Problem 3:
How many binary strings of length ≤ 5 are accepted by the below DFA?



Problem 4:
How many non-isomorphic groups of four elements exist?
How many non-isomorphic groups of five elements exist?

Problem 5:
Find the maximal flow size in the following graph.