ORMC Intermediate 2B: FE Additional Exercises

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Problem 17. *Denote by S the set of whole numbers bigger than 2. Find all functions $f : S \to S$ (f takes element of S as input and gives an element of S as output) satisfying $f(x)f(y) = f(x^2y^2)$ for all $x \neq y$ in S.*

Problem 18. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(xf(y)) - x = f(xy)$.

Problem 19. *Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *(*f *takes a positive real number as input and gives back a positive real number) satisfying* $f(2x + 2f(y)) = x + f(x) + 2y$.

Problem 20. *Suppose $f : \mathbb{R} \to \mathbb{R}$ (note that this is different from $f : \mathbb{Q} \to \mathbb{Q}$) satisfies the Cauchy functional equation. It's known that if you can find an interval $[a, b]$ where f is either bounded below or above on $[a, b]$ then f is linear. Also, f is called a* ***field automorphism*** *of $\mathbb{R}$ if it satisfies $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$. Show using the above fact that all field automorphisms of $\mathbb{R}$ satisfying the Cauchy functional equation are linear.*

Problem 21. *Find all functions* $f : \mathbb{Q} \to \mathbb{Q}$ *satisfying* $f(w) + f(z) = f(x) + f(y)$ *for all equally-spaced inputs* $w < x < y < z$*. By equally spaced, I mean that* $z - y = y - x = x - w$*.*

Problem 22. *For this problem, you need to know that if the limit as x approaches y of $\frac{f(x)-f(y)}{x-y}$ is 0 then f is constant. This limit is called the "derivative" of f. Don't worry about exactly what a limit is rigorously; just play with the equation and try to use the above result. Show that all functions $f : \mathbb{Q} \to \mathbb{Q}$ with $|f(x) - f(y)| \leq (x - y)^2$ are constant.*