Induction, Negation, and Contradiction

Problem 1. *Last class, we proved that we can tile any wall of size $2^n \times 2^n$ with a corner removed using L shaped tiles. The success of these L shaped tiles has helped Clear Skies Ceramics gain interest from many potential customers. However, a lot of these potential customers want the missing tile in their walls to be in different locations. To keep the business booming, prove that L shaped tiles can be used to tile any wall of size $2^n \times 2^n$ with any single tile removed. (Hint: try starting with the small cases. This should look similar to your previous tiling proof.)*

Problem 2. *As a growing business, Clear Skies Ceramics has decided to expand into circular mosaics. However, initial market research has revealed that circular mosaic customers are very peculiar: they only want designs with straight lines cutting across the circle, but never intersecting with more than one other line at a single point, and with neighboring regions formed by this design to have different colors. Given that Clear Skies Ceramics only has two colors of paint, is the expansion into this new market viable?*
(Hint: try adding in lines one by one.)

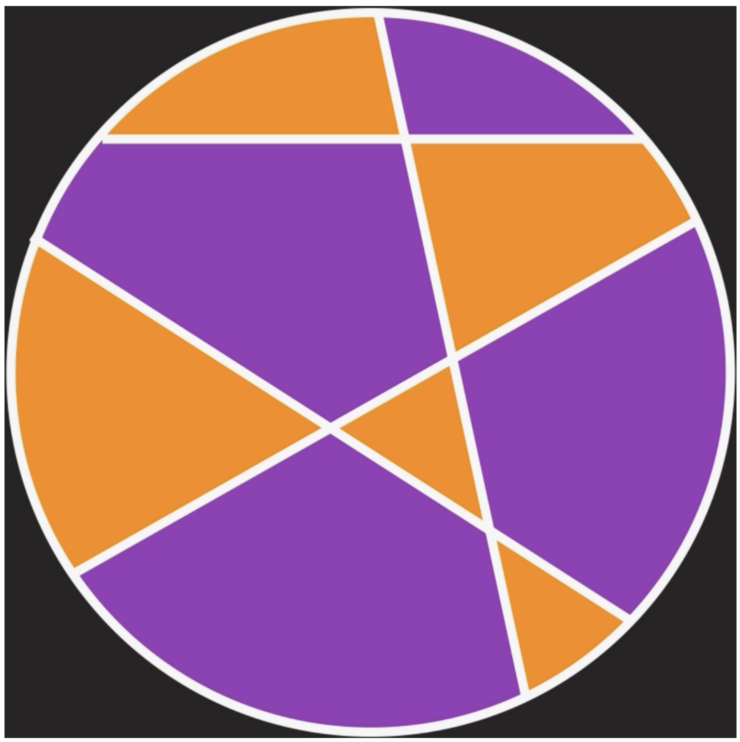

Figure 1: A valid circular mosaic that could be sold.

Problem 3. *Think back to the Tower of Hanoi puzzle you played around with at the beginning of class. What is the smallest number of steps you need to move 2 rings from one side to the other? What about 3 rings? 4? 11?*
Prove the following statement about the puzzle: "The solution with minimal moves to the Tower of Hanoi with n *rings requires* $2^n - 1$ *steps."*

What is a Contradiction?

Last time, we learned how to use induction to prove statements that hold for arbitrarily many cases. We needed this technique to avoid proving similar statements over and over again to infinity and beyond. Today, we'll be learning another proof technique: proof by contradiction.

Sometimes instead of directly proving something is true, it is easier to prove that it cannot be false. A statement being not false is equivalent to it being true. But how can we show without a shadow of a doubt that something must be false?

One approach is to play devil's advocate. If we suppose that something is false, then we've established a logical baseline. From there, if we can find our way toward a logical contradiction, then our original assumption must have been faulty. This search for impossibility is one way to show that something must be false.

Proof by contradiction is the formalization of this idea. Let's dive into the nitty-gritty of how this actually works!

Negating Simple Statements

To prove whether a statement is true or false, we must first agree on what can constitute a statement. Formally, a **mathematical statement** is something that can either be (objectively) true or be (objectively) false. For example, "There are exactly 3 prime numbers in existence" is a perfectly good mathematical statement, whether or not it is actually true!

Problem 4. *Circle the sentences that are mathematical statements.*

1. *13 divides evenly into 26.*
2. *Orange is the best color.*
3. *Meryl Streep currently has 5 Academy Awards.*
4. *π is a rational number.*
5. *Buy me a ticket to The Super Mario Bros. Movie for this Friday.*

The first step in a proof by contradiction is to assume that our initial statement is false. This is equivalent to assuming that the opposite of the statement is true. But what does the opposite of a statement even mean? Formally, the opposite, or **negation**, of a statement is a statement that is true exactly when the original statement is false, and vice versa. That is, exactly one of a statement and its negation must be true. For example, the negation of "I like pies" would be "I don't like pies"; clearly, one of these statements must be true, but it doesn't make sense for both of these statements to be true at once.

Problem 5. *Find the logical negations to the following statements:*

1. *I am older than my brother.*

2. *5 is an odd number.*

3. *It is hotter than 90 degrees outside.*

4. *Anna has at least as many pencils as Jeff*

The sentences in Problem 5 are called **simple statements** because they only have one verb. To negate simple statements, we simply negate the verb!

Negating Compound Statements

Now that we know what simple statements are and how to negate them, what about more complicated statements? **Compound statements** are when you combine multiple simple statements using the words "and" and "or". The combination of two statements using "and" is true exactly when both statements are true. The combination of two statements using "or" is true exactly when at least one of the statements is true. This should all be a review of our Boolean logic packets from last year!

Problem 6. *Fill out the following truth tables.*

Today is cold	*Today is rainy*	*Today is cold AND (today is) rainy*
T	T	
F	T	
T	F	
F	F	

Today is cold	*Today is rainy*	*Today is cold OR (today is) rainy*
T	T	
F	T	
T	F	
F	F	

Just like we could negate simple statements, we can also negate compound statements! We can write the negation of "today is cold and rainy" as "it is not the case that today is cold and rainy". But, when exactly is this negated statement true? If it is not cold and also not rainy today, then obviously it is true that "it is not the case that today is cold and rainy". But what about if it is only cold? Or what if it is only rainy?

Problem 7. *Fill out the following truth table.*

Today is cold	*Today is rainy*	*It is not the case that today is cold and rainy*
T	T	
F	T	
T	F	
F	F	

So, can we write the statement "it is not the case that today is cold and rainy" as a combination of the negated statements "today is not cold", "today is not rainy"?

Problem 8. *Fill out the following truth tables.*

Today is cold	*Today is rainy*	*Today is not cold AND today is not rainy*
T	*T*	
F	*T*	
T	*F*	
F	*F*	

Today is cold	*Today is rainy*	*Today is not cold OR today is not rainy*
T	*T*	
F	*T*	
T	*F*	
F	*F*	

Problem 9. *Based on these truth tables, what is the negation of "A AND B"? Here, A and B are symbols for two simple statements.*

You can use truth tables in the same way to figure out how to negate a compound statement that uses "or".

Problem 10. *What is negation of "A OR B"? Here, A and B are symbols for two simple statements.*

Hint: use truth tables with the same example statements A = "today is cold", B = "today is rainy".

Quantifier Statements

Another way that statements can be made more complex is with **quantifiers**. Quantifiers tell the reader *how many* things the statement is true for. There are two most common types of quantifiers: **universal** and **existential**.

Universal quantifiers are words used to mean that the statement holds for every object that is being referenced, such as the words "all", "every", "any", and "each". An example of a true universal statement is "every person is younger than 200 years old"; an example of a false universal statement is "every natural number is even". On the other hand, existential quantifiers are used to mean that the statement holds for at least one object that is being referenced. The phrases "there exists", "there is (at least one)", and "for some" are all existential quantifiers. An example of a true existential statement is "there is someone that is 50 years old right now"; an example of a false universal statement is "there exists an integer between 0 and 1".

Problem 11. *Circle the statements that use a universal quantifier. Box the statements that use an existential quantifier. Finally, underline the statements that don't use a universal or existential quantifier.*

1. *Any integer is evenly divisible by the number 1.*
2. *Meryl Streep currently has 5 Academy Awards.*
3. *There exists an irrational number greater than 7.*
4. *All squares have an area of exactly 25 centimeters squared.*
5. *There is a planet revolving around the sun.*
6. *The Pythagorean theorem is true.*

The cool thing is that to negate a statement with a universal quantifier, you just replace the universal quantifier with an existential quantifier and negate the rest of the statement! This is because a statement about *all things* is false exactly when it does not hold *for at least one thing*. Similarly, to negate a statement with an existential quantifier, you just replace the existential quantifier with a universal quantifier and negate the rest of the statement.

Problem 12. *Negate each of the following statements. The first two statements are done as examples for you.*

1. *Any integer is evenly divisible by the number 1.*

   There is some integer that is not evenly divisible by the number 1.

2. *There exists an irrational number greater than 7.*

   Every irrational number is less than or equal to 7.

3. *Every triangle has three sides.*

4. *There is a planet revolving around the sun.*

5. *For all integers x, it is true that* $(3x \mod 3) = 0$.

6. *Someone is older than Professor Oleg.*

Opposite Day

Problem 13. *Today is Opposite Day. Isaac and John decide to be very clever and hold a conversation only saying the exact opposite of what they mean. Can you translate what they are saying without using the word "not"?*

Isaac: My day is not good.

John: There is a cloud outside!

Isaac: I have not done any of the problems on my homework!

John: I have done all of the problems on my homework.

Isaac: I think that it is less than 75 degrees outside.

John: Everybody will show up to math circle today.

Isaac: I agree. There will be at least one person that is missing.

Proof by Contrapositive

Consider the statement

> "If it is raining outside, then Dora will open her umbrella."

This is a "if A, then B" type of statement. The rule of contrapositive says that this statement is the same thing as "if not B, then not A". So, how can we rephrase the above statement in this way?

> "If Dora has not opened her umbrella, then it is not raining outside."

We call this reversed statement a contrapositive. A good way to remember contrapositive is with a picture.

$$\begin{tikzcd} A \arrow[rr] & & B \\ & {} \arrow[d, phantom, "\Updownarrow"] & \\ \textbf{not } B & & \textbf{not } A \arrow[ll] \end{tikzcd}$$

Problem 14. *Explain why the statement "If you are Math Circle instructor, then you are at least 18 years old" is logically equivalent to "If you are not yet 18 years old, then you are not a Math Circle instructor".*

Problem 15. *Explain why the statement "If A is true, then B is true" is logically equivalent to "If B is false, then A is false".*

Problem 16. *Can you write the contrapositive of the following statements?*

1. *If it is sunny, then it is not raining*

2. *If every swan is white, then no swan is black*

3. *If it is late, then you need to go to sleep*

4. *If you don't eat food, then you will feel hungry*

Problem 17 (A proof using the contrapositive)**.** *Proving the contrapositive of a statement can sometimes be easier than proving the statement itself. For instance, let us try to prove "If x^2 is even, then x is even".*

1. *What is the contrapositive of the statement "If x^2 is even, then x is even"?*

2. *Use* $\bmod 2$ *arithmetic to prove that the square of an odd number is odd.*

3. *Conclude that if x^2 is even, then x is even.*

Proof by Contradiction

We've finally established the necessary groundwork to write proofs by contradiction. The best way to see how is by example. Let us prove that $\sqrt{2}$ is irrational.

Recall that a number is irrational if it is not rational, i.e., it cannot be represented as an integer fraction. To prove that $\sqrt{2}$ is irrational, we first assume the opposite: that it can be represented as an integer fraction. If we can subsequently show that it can't be represented as an integer fraction, our original assumption must be incorrect. This is because it can't be true that $\sqrt{2}$ is both an integer fraction and not an integer fraction.

Problem 18 (Proof that $\sqrt{2}$ is irrational)**.** *Assume (for contradiction) that $\sqrt{2} = \frac{a}{b}$ for some integers a and b where $b \neq 0$.*

1. *Show that it is possible to choose a and b so that only one of them is divisible by 2.*

2. *We then have that $2 = \frac{a^2}{b^2}$. Show that a is even.*

3. *Show that 4 divides a^2.*

4. *If* 4 *divides* a^2*, show that* 2 *divides* b^2*.*

5. *If* 2 *divides* b^2*, is* b *even or odd?*

6. *Why is this a contradiction? Conclude that there are no integers* a *and* b *so that* $\frac{a}{b} = \sqrt{2}$*.*

Problem 19. *Why doesn't this proof work when we try to show that* $\sqrt{4}$ *is irrational?*

Problem 20. *Can you use the same kind of proof to show that* $\sqrt{3}$ *is irrational?*

One of the classic proofs of mathematics is that there is an infinite number of prime numbers. It isn't clear to me how to directly show the infinitude of primes. Instead, to prove this statement, we can show that it is not possible for there to be a finite number of primes. Proof by contradiction!

Problem 21 (Proof of infinitely many primes)**.**

1. *If we want to prove that there are infinitely many primes, what should we instead assume for our contradiction?*

2. *With our assumption, why is it that we can list all of the primes:*

$$p_1, p_2, \ldots, p_n$$

   Let us now consider the number $p_1 \times p_2 \times \cdots \times p_n + 1$

3. *Does* p_1 *divide* $p_1 \times p_2 \times \cdots \times p_n + 1$*? (Hint: look at the remainder)*

4. *Do any of* $p_1, p_2, \ldots, p_n$ *divide* $p_1 \times p_2 \times \cdots \times p_n + 1$*?*

5. *What can we conclude about $p_1 \times p_2 \times \cdots \times p_n + 1$? Why is this a contradiction? (Hint: Does every integer have a prime number expansion?)*

Problem 22. *Suppose we know that $p_1, p_2, \ldots, p_n$ are primes. Does the previous proof show that $p_1 \times p_2 \times \cdots \times p_n + 1$ must be a prime? Why or why not?*

Paradoxes

Sometimes it can be hard to determine if a statement is true or false. In fact, sometimes, it is impossible to tell if a statement is true or false. Take for instance the statement

> "This statement is false."

This seems troublesome. Why?

Problem 23. *Sam is very pleased with his most recent purchase, a shiny book called the Complete Non-Self-Reference Reference (CNSRR). The ads tell us that this book mentions exactly all the books that don't mention themselves. For instance, Harry Potter and the Sorcerer's Stone doesn't mention itself, as a book, anywhere in the text, so CNSRR mentions it. On the other hand, the dictionary does mention itself, so it does not get mentioned in CNSRR. When Emily hears about Sam's new purchase, she gets worried. "I think there's a problem with your new book," she says. What does Emily mean? What's the problem?*

Problem 24. *Pablo, Gary, Sara, Mark, and Wanda were playing clue last week. Each one was playing as one of five characters: Professor Plum, Mr. Green, Miss. Scarlet, Colonel Mustard or Mrs. White. Everybody but one person committed a crime in a certain room with a certain weapon. You interview the suspects, but quickly realize that they only tell lies. Using these clues, can you solve the mystery?*

Interview Transcript

1. *Pablo: There is a person who is not one of the following:*
   (a) *Pablo*
   (b) *The woman who is playing Colonel Mustard*
   (c) *The man who used Wrench*
   (d) *The Kitchen criminal*
   (e) *The one who is innocent*
2. *Gary: At least one player's name did start with the same letter of the character they were playing.*
3. *Sara: Wanda did not steal the Knife*
4. *Mark: The person who isn't Gary or the person who isn't Sara were not in the Ballroom and not in Study*
5. *Wanda: Mrs. White did not use the Lead Pipe or Professor Plum did not use the Candlestick*
6. *Pablo: The murderer did not have the Rope*
7. *Gary: Mark was not seen in the Cellar with the Candlestick*
8. *Sara: Whoever broke the Wrench did not do it in the Ballroom*