LAMC Junior Circle **September 30, 2012**

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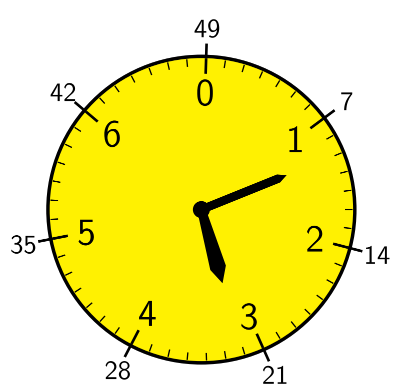
Warm-up

Problem 1 *A water lily grows in the center of a circular pond. The lily doubles in size every day. It takes the flower 100 days to cover the entire pond. How long does it take the lily to cover a half of the pond?*

Clock Arithmetic

The planet of Heptadium in a galaxy far, far away makes one full rotation around its axis in 7 heptahours. The folks inhabiting Heptadium have heptahour clocks similar to the one pictured below.



They further divide a heptahour into 49 heptaminutes and a heptaminute into 49 heptaseconds. The heptahours are marked on the inside of the dial, the heptaminutes – on the outside.

Problem 2 *What time does the clock show?*

Problem 3 *One Heptadian tells another, "The next day will begin in one minute." What time is his watch showing?*

On the dial of the heptaclock, 0 coincides with 7. We write this fact down as

$$7 \equiv 0 \ (mod\ 7) \tag{1}$$

and read it as *7 is congruent to 0 modulo 7.* The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The *mod* 7 symbol tells us that the circle is divided into 7 equal parts, so 7 coincides with 0, 8 with 1, 9 with 2, and so on. Or in the new notations,

$$8 \equiv 1 \ (mod\ 7),\ \ 9 \equiv 2 \ (mod\ 7), \ldots, 13 \equiv 6 \ (mod\ 7),$$

$$14 \equiv 7 \equiv 0 \ (mod\ 7).$$

Problem 4

$$22 \ (mod\ 7) \equiv$$

$$100 \ (mod\ 7) \equiv$$

$$6 + 5 \equiv \qquad\qquad (mod\ 7)$$

Note that the notation $n \ (mod\ 7)$ represents not a single number, but all the numbers of the form

$$\ldots, n - 3 \cdot 7, n - 2 \cdot 7, n - 7, n, n + 7, n + 2 \cdot 7, n + 3 \cdot 7, \ldots$$

The infinite set

$$n \ (mod\ 7) = \{n, n \pm 7, n \pm 2 \cdot 7, n \pm 3 \cdot 7, \ldots\}$$

is called a *congruence class*. For example,

$$2 \ (mod\ 7) = \{\ldots, -19, -12, -5, 2, 9, 16, 23, \ldots\}.$$

Problem 5 *Are the classes* $0 \ (mod\ 7)$ *and* $7 \ (mod\ 7)$ *the same? Why or why not?*

Problem 6 *Write down five different representatives for each of the following two congruence classes.*

$$4 \ (mod\ 7) \equiv \qquad\qquad 6 \ (mod\ 7) \equiv$$

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Divide each of the ten numbers by seven. Compare the remainders.

Problem 7 *How many different remainders for division by 7 are there?*

Problem 8 *How many different mod 7 congruence classes are there? Why?*

Problem 9 *Find the standard (clock-face) representatives of the following classes.*

$$1000 \ (mod\ 7) \equiv$$

$$7000 \ (mod\ 7) \equiv$$

$$8000 \ (mod\ 7) \equiv$$

Can you solve the last two problems without using division?

Problem 10 *An experiment in a Heptadium nuclear lab starts at 4:00 and runs for 2000 hours. What time will it end?*

To count minutes and seconds, the Heptadians need the *mod* 49 arithmetic, that of a circle divided into 49 equal parts.

Problem 11 *Find the standard (clock-face) representatives of the following classes (the problem continues to the next page).*

$$50 \ (mod \ 49) \equiv$$

$$500 \ (mod \ 49) \equiv$$

$$40 + 30 \equiv \qquad\qquad (mod \ 49)$$

$$-14 \ (mod \ 49) \equiv$$

$$7 \times 7 \equiv \qquad\qquad (mod \ 49)$$

$$7 \times 10 \equiv \qquad\qquad (mod\ 49)$$

Problem 12 *They run four experiments in a Heptadium biological lab. The first three take an equal amount of time, the last experiment is as long as the first three together. The experiments are run one after another without time gaps. The first begins at 1:00. The last ends at 2:00. The first experiment takes more than a day, but less than two days and lasts a whole number of hours. How long does the last experiment take?*

Problem 13 *Find the standard (clock-face) representatives of the following classes (the problem continues to the next page). When solving the division problems, remember: division and multiplication are opposite operations. Division undoes what multiplication does.*

$$9 \ (mod\ 7) \equiv$$

$$-10 \ (mod\ 7) \equiv$$

$$-100 \ (mod\ 7) \equiv$$

$$1 - 3 \equiv \qquad\qquad (mod\ 7)$$

$$4 + 4 \equiv \qquad\qquad (mod\ 7)$$

$$2 \times 4 \equiv \qquad\qquad (mod\ 7)$$

$$1 \div 2 \equiv \qquad\qquad (mod\ 7)$$

$$1 \div 4 \equiv \qquad\qquad (mod\ 7)$$

$$2 \div 3 \equiv \qquad\qquad (mod\ 7)$$

$$4 \div 5 \equiv \qquad\qquad (mod\ 7)$$

Recall that

$$a^b = \underbrace{a \times a \times \ldots \times a}_{b\ times}$$

Problem 14 *Find the following number.*

$$3^{100}\ (mod\ 7) \equiv$$

Problem 15 *Find the following number.*

$$5^{1234} \ (mod\ 7) \equiv$$

Problem 16 *Find the last two digits of the following number.*

$$7^{2012}$$