ORMC AMC 10/12 Training Week 2
Graph Theory

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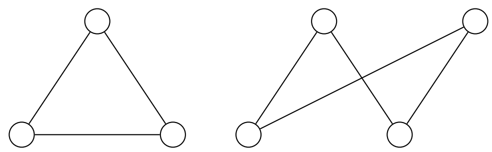
1 Definitions

A *graph*[1] G is a collection V of vertices and a collection E of edges, each connecting two vertices. When both E and V are finite, G is called a *finite graph*. All graphs we consider hereafter will be finite.



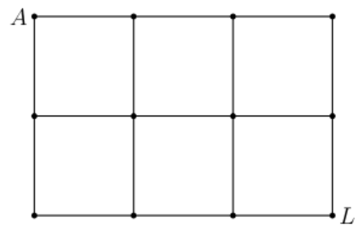
1. A graph is *simple* if no edge starts and ends at the same vertex, and there is at most one edge between any pair of vertices. All graphs we consider hereafter will be simple.
2. Two vertices v and w are *adjacent* if there is an edge joining v and w. An edge and a vertex are *incident* if the vertex is an endpoint of the edge.
3. The *degree* of a vertex v in a graph G is the number of edges connecting to v, denoted by $d(v)$.
4. A *path* is a graph G is a finite sequence of vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}. The number t of edges is the *length* of the path.
5. A *cycle* is a path with $v_t = v_0$.
6. A graph is *connected* if for every pair of vertices v and w, there is a path from v to w. A graph is *disconnected* if it is not connected.
7. Let $G = (V, E)$ be a graph. The *complement* of G, denoted by $\bar{G}$, is the graph with the same set V of vertices, and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.
8. The *complete graph* on n vertices, denoted by K_n, is the graph with n vertices such that there is an edge connecting every pair of distinct vertices.
9. A graph is *bipartite* if its vertices V can be partitioned into two disjoint subsets V_1 and V_2 such that no edge has both endpoints within the same set. A graph is *complete bipartite* if it is bipartite and all possible edges between V_1 and V_2 are drawn. The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.
10. A graph is *planar* if it can be drawn on a plane such that no two edges intersect (except at the vertices). Convince yourself that Euler's formula $V + F - E = 2$ for polyhedrons also holds for planar graphs.

[1] Some definitions and exercises in this worksheet are drawn from http://web.mit.edu/yufeiz/www/imo2008/tang-graph.pdf

2 Exercises

2.1 Warm-up Problems

Exercise 1 (2019 AMC 12B #10)**.** The figure below is a map showing 12 cities and 17 roads connecting certain pairs of cities. Paula wishes to travel along exactly 13 of those roads, starting at city A and ending at city L, without traveling along any portion of a road more than once. (Paula is allowed to visit a city more than once.) How many different routes can Paula take?



Exercise 2. Prove that, among any group of 6 people, either there are 3 people that know each other, or there are 3 people that do not know each other.

Exercise 3 (Handshaking Lemma)**.** A group of people at a party is shaking hands. Prove that the sum of the number of people that each person shakes hand with is twice the total number of handshakes happened.

Exercise 4. Let G be a disconnected graph. Prove that its complement $\bar{G}$ is connected.

Exercise 5. Prove that a graph is bipartite if and only if it does not contain an odd cycle.

Exercise 6. Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?

Exercise 7. Consider a polyhedron with at least five faces such that exactly three edges emerge from each vertex. Two players play the following game: the players sign their names alternately on precisely one face that has not been previously signed. The winner is the player who succeeds in signing the name on three faces that share a common vertex. Assuming optimal play, prove that the player who starts the game always wins.

2.2 Exercises

Exercise 8 (2021 AMC 12B #20)**.** A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

Exercise 9 (2012 AIME II #14)**.** In a group of nine people each person shakes hands with exactly two of the other people from the group. Let N be the number of ways this handshaking can occur. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement. Find the remainder when N is divided by 1000.

Exercise 10. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column

Exercise 11. There are 10 points in $\mathbb{R}^3$ such that no three are on the same line and no four are on the same plane. What is the maximum number of edges that can be drawn among these points such that they do not form triangles or quadrilaterals?

Exercise 12. Let G be a connected graph with an even number of vertices. Prove that you can select a subset of edges of G such that each vertex is incident to an odd number of the selected edges.

Exercise 13 (USAMO 2007)**.** An animal with n cells is a connected figure consisting of n equal-sized cells. A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

2.3 Advanced Problems

Exercise 14 (Turán)**.** A simple graph G has n vertices. Prove that if G has more than $n^2/4$ edges, then G contains a triangle, i.e., 3 vertices such that each two of them are connected by an edge.

Exercise 15 (Italy 2007)**.** Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to color the computers and cables such that no two computers have the same color, no two cables joined to a common computer have the same color, and no computer is assigned the same color as any cable joined to it. Prove that this can be done using n colors.

Exercise 16 (USAMO 2023 Problem 3)**.** Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find all possible values of $k(C)$ as a function of n.

Exercise 17. 20 football teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

Exercise 18. Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n.