In our first problem, we recall some concepts about real projective geometry. Classically, we do geometry in an affine space $\mathbb{R}^n$. However, as we will see throughout this worksheet, working with a projective space $\mathbb{P}_\mathbb{R}^n$ has some advantages.

The projective space $\mathbb{P}_\mathbb{R}^n$ consists of $(n+1)$-tuples $[x_0 : \cdots : x_n]$ where not all the entries $x_i$ are equal to zero. Two points $[x_0 : \cdots : x_n]$ and $[y_0 : \cdots : y_n]$ are said to be equal in $\mathbb{P}_\mathbb{R}^n$ if there exists a constant $\lambda \neq 0$ for which $x_i = \lambda y_i$ for each $i$. For instance, in $\mathbb{P}_\mathbb{R}^2$ we have $[3 : 3 : 3] = [1 : 1 : 1]$. The number $n$ is called the dimension of the real projective space.

In the first problem, we will determine whether points in projective spaces are equal or not.

**Problem 1.0:** Are the points $[0 : 1]$ and $[0 : 2]$ equal in the projective space $\mathbb{P}_\mathbb{R}^1$?
Are the points $[0 : 1]$ and $[1 : 1]$ equal in the projective space $\mathbb{P}_\mathbb{R}^1$?
Are the points $[1 : 2 : 3]$ and $[3 : 6 : 9]$ equal in the projective space $\mathbb{P}_\mathbb{R}^2$?

**Solution 1.0:**
Problem 1.1: How many points does the 0-dimensional projective space have? Meaning, how many points does \( \mathbb{P}^0_\mathbb{R} \) have? Consider the function:

\[
\phi: \mathbb{R}^1 \rightarrow \mathbb{P}^1_\mathbb{R} \quad \phi(x) := [x : 1].
\]

Is the function \( \phi \) one-to-one? Is the function \( \phi \) onto? What points does \( \phi \) miss?

Solution 1.1:
Problem 1.2: Consider the function:

\[ \phi : \mathbb{R}^n \to \mathbb{P}_\mathbb{R}^n \quad \text{given by} \quad \phi(x_1, \ldots, x_n) = [x_1 : \cdots : x_n : 1]. \]

Is the function \( \phi \) one-to-one?
Is the function \( \phi \) onto?
If not, what points does \( \phi \) miss?

Solution 1.2:
In mathematics, we write
\[ X = A \sqcup B, \]
if the set \( X \) is the disjoint union of \( A \) and \( B \). This means that every element of \( X \) lies either in \( A \) or \( B \) but no element of \( X \) lies in both \( A \) and \( B \). In symbols, we can write \( X = A \cup B \) and \( A \cap B = \emptyset \).

For instance, we can write
\[ \mathbb{R} := \mathbb{R}^- \sqcup \{0\} \sqcup \mathbb{R}^+, \]
where \( \mathbb{R}^- \) denotes strictly negative numbers and \( \mathbb{R}^+ \) denotes strictly positive numbers. However, the equality
\[ \{0, 1, 2\} = \{0, 1\} \sqcup \{0, 2\} \]
is not correct, as both sets contain zero. In this case, it is only correct to write
\[ \{0, 1, 2\} = \{0, 1\} \cup \{0, 2\}. \]

**Problem 1.3:** From the previous problems, deduce that we can write
\[ \mathbb{P}^n_{\mathbb{R}} = \mathbb{R}^n \sqcup \mathbb{P}^{n-1}_{\mathbb{R}}. \]
Iterating the previous equality, conclude that there is an equality:
\[ \mathbb{P}^n_{\mathbb{R}} = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0. \]
The points in \( \mathbb{P}^n_{\mathbb{R}} \) for which the last component \( x_n \) is zero, are called points at infinity.

**Solution 1.3:**
A projective line in $\mathbb{P}^2_\mathbb{R}$ is the set of points $[x_0 : x_1 : x_2]$ satisfying a linear equation
\[ \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0, \]
where the $\lambda_i$ are real parameters.

As we discussed above, all the points in $\mathbb{P}^2_\mathbb{R}$ whose last coordinate is 1 can be considered as points in $\mathbb{R}^2$. The point $[x_0 : x_1 : 1]$ corresponds to $(x_0, x_1) \in \mathbb{R}^2$. The intersection of a line with $\mathbb{R}^2$ is called the affine part of the line. The intersection of the line with the line at infinity are called the points at infinity.

For instance, the line $x_0 + x_1 + x_2 = 0$, in $\mathbb{P}^2_\mathbb{R}$, has affine part $(t, t - 1)$ in $\mathbb{R}^2$ and the point $[1 : -1 : 0]$ at infinity.

**Problem 1.4:** For the following projective lines, find their affine parts and their points at infinity.

- The projective line $x_0 = 0$ in $\mathbb{P}^2_\mathbb{R}$.
- The projective line $x_0 = 1$ in $\mathbb{P}^2_\mathbb{R}$.
- The projective line $x_0 + 2x_1 + x_2 = 0$ in $\mathbb{P}^2_\mathbb{R}$.
- The projective line $x_2 = 0$ in $\mathbb{P}^2_\mathbb{R}$. This projective line is called the line at infinity as all its points are at infinity.

**Solution 1.4:**
Two projective lines in $\mathbb{P}_R^2$ are said to have an \textit{affine intersection point}, if they share a common point in $\mathbb{R}^2$. Otherwise, we say that the lines intersect at infinity.

**Problem 1.5:** For the following pairs of lines, decide whether they have an affine intersection point or they intersect at infinity. In any case, find the intersection point.

- The line $x_0 + x_1 + x_2 = 0$ and the line $x_0 + 2x_1 + x_2 = 0$.
- The line $x_0 - x_1 + x_2 = 0$ and the line $x_0 - x_1 + 2x_2 = 0$.
- The line $3x_0 - 2x_1 + 4x_2 = 0$ and the line $3x_0 - 2x_1 + 5x_2 = 0$.

**Solution 1.5:**
**Problem 1.6:** Show that two projective lines in \( \mathbb{P}_r^2 \) always intersect: either at an affine point or at infinity. Can two lines intersect both at infinity and at an affine point?

**Solution 1.6:**
Problem 1.7: Consider the projective line $\ell_1$ given by the equation 

$$x_0 + x_1 + x_2 = 0$$

in the projective space $\mathbb{P}_R^2$. Find all the projective lines $\ell_2$ for which $\ell_1$ and $\ell_2$ intersect only at infinity.

Solution 1.7:
Three points in $\mathbb{P}_R^2$ are said to be *collinear* if they lie in a projective line. For instance, the points $[1 : -1 : 0], [3 : -4 : 1], \text{ and } [0 : 1 : -1]$ are distinct points in $\mathbb{P}_R^2$ that are collinear. Indeed, they all lie in the projective line $x_0 + x_1 + x_2 = 0$.

**Problem 1.8:** Let $(a_0, a_1), (b_0, b_1), (c_0, c_1)$ be three collinear points in $\mathbb{R}^2$. Show that the points $[a_0 : a_1 : 1], [b_0 : b_1 : 1], \text{ and } [c_0 : c_1 : 1]$ are collinear in $\mathbb{P}_R^2$.

**Solution 1.8:**
In $\mathbb{R}^2$, there are 4 possible ways that three lines can intersect, depending on whether the lines are parallel or not. This is described in the following picture:

\[
\begin{array}{c}
\times \\
\equiv \\
\neq \\
\triangle
\end{array}
\]

In $\mathbb{P}_R^2$ there are no parallel lines, so there are only two possibilities.

**Problem 1.9**: Show that three projective lines $\ell_1, \ell_2$, and $\ell_3$ in $\mathbb{P}_R^2$ either satisfy:

- Their intersection is non-empty, i.e., $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$, or
- the three lines form a triangle, i.e., $\ell_1 \cap \ell_2 = p, \ell_2 \cap \ell_3 = q$, and $\ell_1 \cap \ell_3 = r$ for three different points $p, q$, and $r$ in the projective space.

For each of the previous situation, find explicit lines $\ell_1, \ell_2$, and $\ell_3$. For each explicit choice of the lines, find all the intersection points.

**Solution 1.9**