OLGA RADKO MATH CIRCLE: ADVANCED 3

JOAQUÍN MORAGA

Spring Quarter - Worksheet 1: Projective Lines

In our first problem, we recall some concepts about real projective geometry. Classically, we do geometry in an *affine space* $\mathbb{R}^n$. However, as we will see throughout this worksheet, working with a *projective space* $\mathbb{P}^n_{\mathbb{R}}$ has some advantages.

The projective space $\mathbb{P}^n_{\mathbb{R}}$ consists of $(n+1)$-tuples

$$[x_0 : \cdots : x_n]$$

where not all the entries x_i are equal to zero. Two points $[x_0 : \cdots : x_n]$ and $[y_0 : \cdots : y_n]$ are said to be equal in $\mathbb{P}^n_{\mathbb{R}}$ if there exists a constant $\lambda \neq 0$ for which $x_i = \lambda y_i$ for each i. For instance, in $\mathbb{P}^2_{\mathbb{R}}$ we have $[3 : 3 : 3] = [1 : 1 : 1]$. The number n is called the *dimension* of the real projective space.

In the first problem, we will determine whether points in projective spaces are equal or not.

Problem 1.0: Are the points $[0 : 1]$ and $[0 : 2]$ equal in the projective space $\mathbb{P}^1_{\mathbb{R}}$?
Are the points $[0 : 1]$ and $[1 : 1]$ equal in the projective space $\mathbb{P}^1_{\mathbb{R}}$?
Are the points $[1 : 2 : 3]$ and $[3 : 6 : 9]$ equal in the projective space $\mathbb{P}^2_{\mathbb{R}}$?

Solution 1.0:

Problem 1.1: How many points does the 0-dimensional projective space have?
Meaning, how many points does $\mathbb{P}^0_{\mathbb{R}}$ have?
Consider the function:

$$\phi\colon \mathbb{R}^1 \to \mathbb{P}^1_{\mathbb{R}} \qquad \phi(x) := [x : 1].$$

Is the function ϕ one-to-one?
Is the function ϕ onto?
What points does ϕ miss?

Solution 1.1:

Problem 1.2: Consider the function:

$$\phi\colon \mathbb{R}^n \to \mathbb{P}^n_{\mathbb{R}} \quad \text{given by} \quad \phi(x_1, \ldots, x_n) = [x_1 : \cdots : x_n : 1].$$

Is the function ϕ one-to-one?
Is the function ϕ onto?
If not, what points does ϕ miss?
Solution 1.2:

In mathematics, we write

$$X = A \sqcup B,$$

if the set X is the *disjoint union* of A and B. This means that every element of X lies either in A or B but no element of X lies in both A and B. In symbols, we can write $X = A \cup B$ and $A \cap B = \emptyset$.

For instance, we can write

$$\mathbb{R} := \mathbb{R}^- \sqcup \{0\} \sqcup \mathbb{R}^+,$$

where $\mathbb{R}^-$ denotes strictly negative numbers and $\mathbb{R}^+$ denotes strictly positive numbers. However, the equality

$$\{0, 1, 2\} = \{0, 1\} \sqcup \{0, 2\}$$

is not correct, as both sets contain zero. In this case, it is only correct to write

$$\{0, 1, 2\} = \{0, 1\} \cup \{0, 2\}.$$

Problem 1.3: From the previous problems, deduce that we can write

$$\mathbb{P}^n_{\mathbb{R}} = \mathbb{R}^n \sqcup \mathbb{P}^{n-1}_{\mathbb{R}}.$$

Iterating the previous equality, conclude that there is an equality:

$$\mathbb{P}^n_{\mathbb{R}} = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0.$$

The points in $\mathbb{P}^n_{\mathbb{R}}$ for which the last component x_n is zero, are called *points at infinity.*
Solution 1.3:

A *projective line* in $\mathbb{P}^2_{\mathbb{R}}$ is the set of points $[x_0 : x_1 : x_2]$ satisfying a *linear equation*

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0,$$

where the λ_i are real parameters.

As we discussed above, all the points in $\mathbb{P}^2_{\mathbb{R}}$ whose last coordinate is 1 can be considered as points in $\mathbb{R}^2$. The point $[x_0 : x_1 : 1]$ corresponds to $(x_0, x_1) \in \mathbb{R}^2$. The intersection of a line with $\mathbb{R}^2$ is called the *affine part* of the line. The intersection of the line with the line at infinity are called the *points at infinity.*

For instance, the line

$$x_0 + x_1 + x_2 = 0,$$

in $\mathbb{P}^2_{\mathbb{R}}$, has affine part $(t, t-1)$ in $\mathbb{R}^2$ and the point $[1 : -1 : 0]$ at infinity.

Problem 1.4: For the following projective lines, find their affine parts and their points at infinity.

- The projective line $x_0 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$.
- The projective line $x_0 = 1$ in $\mathbb{P}^2_{\mathbb{R}}$.
- The projective line $x_0 + 2x_1 + x_2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$.
- The projective line $x_2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. This projective line is called *the line at infinity* as all its points are at infinity.

Solution 1.4:

Two projective lines in $\mathbb{P}^2_{\mathbb{R}}$ are said to have an *affine intersection point*, if they share a common point in $\mathbb{R}^2$. Otherwise, we say that the lines intersect at infinity.

Problem 1.5: For the following pairs of lines, decide whether they have an affine intersection point or they intersect at infinity. In any case, find the intersection point.

- The line $x_0 + x_1 + x_2 = 0$ and the line $x_0 + 2x_1 + x_2 = 0$.
- The line $x_0 - x_1 + x_2 = 0$ and the line $x_0 - x_1 + 2x_2 = 0$.
- The line $3x_0 - 2x_1 + 4x_2 = 0$ and the line $3x_0 - 2x_1 + 5x_2 = 0$.

Solution 1.5:

Problem 1.6: Show that two projective lines in $\mathbb{P}_r r^2$ always intersect: either at an affine point or at infinity. Can two lines intersect both at infinity and at an affine point?

Solution 1.6:

Problem 1.7: Consider the projective line ℓ_1 given by the equation

$$x_0 + x_1 + x_2 = 0$$

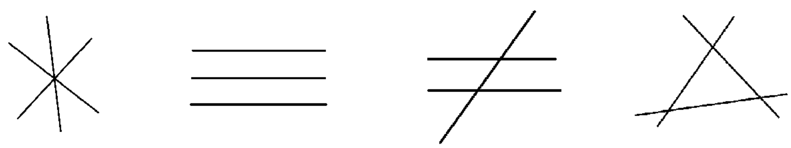
in the projective space $\mathbb{P}^2_{\mathbb{R}}$. Find all the projective lines ℓ_2 for which ℓ_1 and ℓ_2 intersect only af infinity.

Solution 1.7:

Three points in $\mathbb{P}^2_{\mathbb{R}}$ are said to be *collinear* if they lie in a projective line. For instance, the points $[1 : -1 : 0], [3 : -4 : 1]$, and $[0 : 1 : -1]$ are distinct points in $\mathbb{P}^2_{\mathbb{R}}$ that are collinear. Indeed, they all line in the projective line $x_0 + x_1 + x_2 = 0$.

Problem 1.8: Let $(a_0, a_1), (b_0, b_1), (c_0, c_1)$ be three collinear points in $\mathbb{R}^2$. Show that the points $[a_0 : a_1 : 1], [b_0 : b_1 : 1]$, and $[c_0 : c_1 : 1]$ are collinear in $\mathbb{P}^2_{\mathbb{R}}$.
Solution 1.8:

In $\mathbb{R}^2$, there are 4 possible ways that three lines can intersect, depending on whether the lines are parallel or not. This is described in the following picture:



In $\mathbb{P}^2_{\mathbb{R}}$ there are no parallel lines, so there are only two possibilities.

Problem 1.9: Show that three projective lines ℓ_1, ℓ_2, and ℓ_3 in $\mathbb{P}^2_{\mathbb{R}}$ either satisfy:

- Their intersection is non-empty, i.e., $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$, or
- the three lines form a triangle, i.e., $\ell_1 \cap \ell_2 = p, \ell_2 \cap \ell_3 = q$, and $\ell_1 \cap \ell_3 = r$ for three different points p, q, and r in the projective space.

For each of the previous situation, find explicit lines ℓ_1, ℓ_2, and ℓ_3. For each explicit choice of the lines, find all the intersection points.

Solution 1.9

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: jmoraga@math.ucla.edu