Induction, Induction, ...

Suppose that you wanted to climb a (possibly infinitely) tall ladder. You start climbing from the first rung. If you knew that climbing to any rung meant that you could climb to the next one, could you climb the entire ladder?

Induction is the formalization of this idea. If you want to prove that a statement is true for every natural number, you can start with the simplest case. If you can show that the statement holding for an arbitrary number n implies that it holds for the next number $n+1$, then the statement must hold for all natural numbers.

Every proof by induction follows a specific pattern. First a statement is proven true for some small, easy to prove case (usually when $n = 1$). This is called the base case. The next step is proving that if a statement is true for the n^{th} step, then it must be true for the $n + 1^{th}$ step as well. This is called the inductive step.

Proof by induction is a powerful tool in a mathematician's kit. Let's use it to show the following equalities:

Problem 1. $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Problem 2. $1 + 3 + \cdots + (2n - 1) = n^2$

Problem 3. $2 + 5 + \cdots + (3n - 1) = \dfrac{3n^2 + n}{2}$

Problem 4. $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Problem 5. *Prove or disprove: For any natural number* x, $x^2 + x + 41$ *is always a prime number.*

Here are two challenge problems if you're feeling spicy. Make an honest attempt, but don't be afraid to skip them if you're stuck!

Problem 6. $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

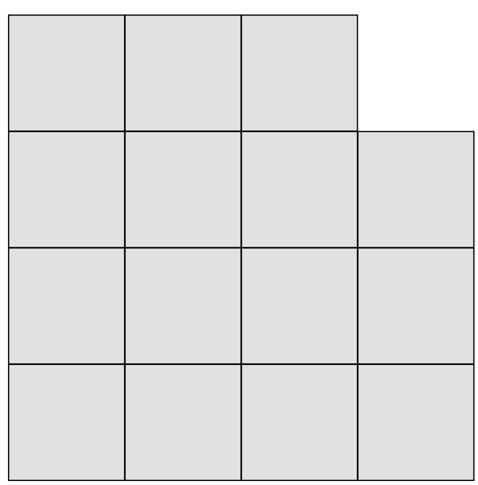
Problem 7. *Show that for any positive integer n, the number consisting of 3^n ones is divisible by 3^n. Eg. $k = 111111111$, is divisible by 9.*

Alright, that's enough of working with stale formulas. Induction can also apply to problems with pictures!

Problem 8. *Suppose that you work at Clear Skies Ceramics, a factory that manufactures beautiful azure blue ceramic tiles. Your factory can make these tiles in two sizes: a 1 inch by 1 inch square, or a 1 inch by 2 inch rectangle.*

Suppose that one section of trim is 1 inch by 2 feet long. How many different ways can you fill the trim with 1 by 1 and 1 by 2 tiles? (Hint: Start small and count the number of tilings for $n = 1, 2, 3$. Do you notice a pattern?)

Problem 9. *You decide to create a new tile by gluing a 1 by 1 and 1 by 2 tile into an L shape. A potential customer wants to buy tiles to fill a 4 inch by 4 inch wall, with a corner removed. Can he do so with just these new L shaped tiles?*



Problem 10. *Your new L shaped tiles are a hit! It turns out that they can be used to tile any wall of size 2^n by 2^n with a corner removed.*

Prove this creative marvel.

Problem 11. *Given a natural number* n*, let* $P(n)$ *be the statement:* n^2+5n+1 *is an even integer. Show that if* $P(n)$ *is true, then* $P(n+1)$ *is true.*

Problem 12. *For what natural numbers* n *is* $P(n)$ *(from Problem 11) true? What is the moral of this problem?*

Problem 13. *Let $x \geq -1$. Prove Bernoulli's inequality for all natural numbers n:*

$$1 + nx \leq (1+x)^n.$$

Problem 14. *Let $a, b > 0$. Prove that for all natural numbers n we have:*

$$a^n + b^n \leq (a+b)^n.$$