ORMC AMC 10/12 Training Week 1

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1 USAJMO 2023

1. Find all triples of positive integers (x, y, z) that satisfy the equation

$$2(x + y + z + 2xyz)^2 = (2xy + 2yz + 2zx + 1)^2 + 2023.$$

2. In an acute triangle ABC, let M be the midpoint of BC. Let P be the foot of the perpendicular from C to AM. Suppose that the circumcircle of triangle ABP intersects line BC at two distinct points B and Q. Let N be the midpoint of AQ. Prove $NB = NC$.

3. Consider an n-by-n board of of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a *maximal grid-aligned* configuration on the board if C consists of $(n^2 - 1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum values of $k(C)$ as a function of n.

4. Two players, B and R, play the following game on an infinite grid of unit squares, all initially colored white. The players take turns starting with B. On B's turn, B selects one white unit square and colors it blue. On R's turn, R selects two white unit squares and colors them red. The players alternate until B decides to end the game. At this point, B gets a score, given by the number of unit squares in the largest (in terms of area) simple polygon containing only blue unit squares. What is the largest score B can guarantee?

   (A *simple polygon* is a polygon (not necessarily convex) that does not intersect itself and has no holes.)

5. A positive integer a is selected, and some positive integers are written on a board. Alice and Bob play the following game. On Alice's turn, she must replace some integer n on the board with $n + a$, and on Bob's turn he must replace some even integer n on the board with $n/2$. Alice goes first and they alternate turns. If on his turn Bob has no valid moves, the game ends.

   After analyzing the integers on the board, Bob realizes that, regardless of what moves Alice makes, he will be able to force the game to end eventually. Show that, in fact, for this value of a and these integers on the board, the game is guaranteed to end regardless of Alice's or Bob's moves.

6. Isoceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Let D be an arbitrary point inside BC such that $BD \neq DC$. Ray AD intersects ω again at E (other than A). Point F (other than E) is chosen on ω such that $\angle DFE = 90°$. Line FE intersects rays AB and AC at points X and Y respectively. Prove that $\angle XDE = \angle EDY$.

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(Problems repeated from USAJMO have been left out)

1. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2.$$

2. The same settings as in USAJMO Problem 3. Find all possible values of $k(C)$ as a function of n.

3. Let $n \geq 3$ be an integer. We say that an arrangement of the numbers $1, 2, \cdots, n^2$ in an $n \times n$ table is *row-valid* if the numbers in each row can be permuted to form an arithmetic progression, and *column-valid* if the numbers in each column can be permuted to form an arithmetic progression. For what values of n is it possible to transform any row-valid arrangement into a column-valid arrangement by permuting the numbers in each row?

4. Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c opposite A, B, and C respectively. Let D be an arbitrary point on the circumcircle of $\triangle ABC$ that does not lie on any of the lines II_a, I_bI_c, or BC. Suppose the circumcircles of $\triangle DII_a$ and $\triangle DI_bI_c$ intersect at two distinct points D and F. If E is the intersection of lines DF and BC, prove that $\angle BAD = \angle EAC$.

3 USAJMO 2023 Solutions

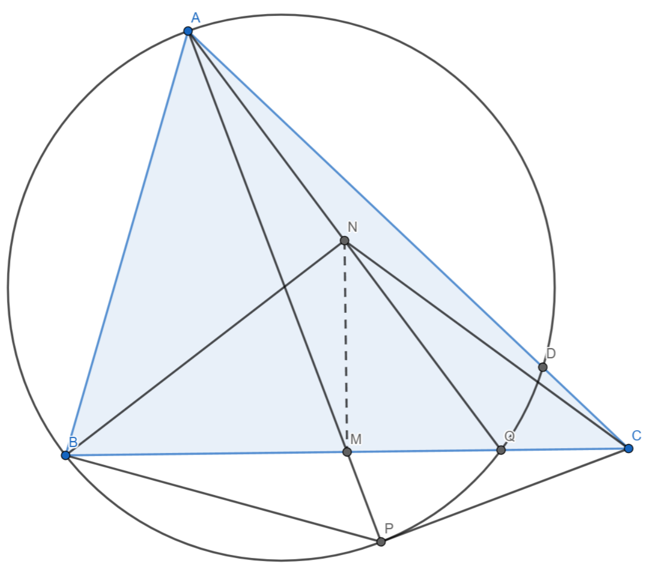
1. $2(x+y+z+2xyz)^2 = (2xy+2yz+2zx+1)^2 + 2023$.

   (a) Factorize

$$\begin{aligned}
2023 &= (\sqrt{2}(x+y+z+2xyz))^2 - (2xy+2yz+2zx+1)^2 \\
&= (\sqrt{2}(x+y+z+2xyz) + 2xy+2yz+2zx+1)(\sqrt{2}(x+y+z+2xyz) - (2xy+2yz+2zx+1)) \\
&= (\sqrt{2}x+1)(\sqrt{2}y+1)(\sqrt{2}z+1)(\sqrt{2}x-1)(\sqrt{2}y-1)(\sqrt{2}z-1) \\
&= (2x^2-1)(2y^2-1)(2z^2-1).
\end{aligned}$$

   (b) $2023 = 7 \cdot 17^2$. All solutions are $(x, y, z) = (2, 3, 3)$ and permutations.

2. The picture is as follows.



   (a) Because M is the midpoint of BC, it suffices to show MN is perpendicular to BC.

   (b) Since $\angle APC = 90^\circ$ and $\angle AMC = \angle BMP$, it suffices to show $\angle AMN = \angle BCP$.

   (c) We know $\angle MAN = \angle CBP$. Then it suffices to show $\triangle AMN \sim \triangle BCP$.

   (d) In a circle we have $\triangle AQM \sim \triangle BPM$. This gives

$$\frac{AQ}{BP} = \frac{MA}{MB}.$$

   Since N and M are midpoints,

$$\frac{AN}{BP} = \frac{MA}{BC}.$$

   Then

$$\frac{AN}{AM} = \frac{BP}{BC},$$

   which gives $\triangle AMN \sim \triangle BCP$.

3. The answer is $(n+1)^2/4$.

   (a) We claim that, starting from a given configuration C_0, any configuration that can be obtained from C_0 is uniquely determined by the place of the empty square.

   (b) Note that $(x, y) \pmod 2$ is invariant during the moves. Then there are at most $(n+1)^2/4$ possible places of the empty square.

   (c) An example that attains $(n+1)^2/4$ is a snake-like configuration.

4. The answer is 4.

   (a) R can move greedily to cover the boundary of blue polygons. Any polygon of area $n < 4$ needs more than $2n$ tiles to cover the boundary, and R can force the 2×2 case.