ORMC Intermediate 2B: Functional Equations

Sanjit Dandapanthula

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Warm-up

Find a number z satisfying $2z^3 + 3z^2 - 5z + 1 = 1$. Then, if $f(x-3) = 2x^3 + 3x^2 - 5x + 1$, write down a formula for the function $f(x)$ (i.e. $f(x) = \cdots$).

Notation

This paragraph is a bit dry, but feel free to come back to it or ask an instructor if any of the symbols used later in the packet are confusing. We use $f : \mathbb{R} \to \mathbb{R}$ to denote a function which takes in a real number and gives back a real number. In this case, we could have $f(\sqrt{2}) = \pi$, for example. Also, we write $f : \mathbb{R}^2 \to \mathbb{R}$ to denote a function which takes in two real numbers and produces a real number as output. For example, we could have $f(0, \sqrt{2}) = 1$. In addition, we write $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ to denote a function which takes in a nonzero real number and gives back a real number. For example, in this case we could have $f(2) = \sqrt{6}$ but $f(0)$ cannot be defined. Finally, we use $f : \mathbb{Q} \to \mathbb{Q}$ to denote a function which takes in a fraction and gives back a fraction. In this case, we could have $f\left(\frac{2}{3}\right) = -1$, for example, but $f(\sqrt{2})$ won't be defined.

1 Introduction

The term **functional equation** describes problems where we want to solve for the function, not the variable. They appear frequently in competition math, as well as in more advanced topics. For example, the geodesic problem from differential geometry is to find the shortest path from one point to another on a surface. This is a functional equation, since we want to find the function (or path) having shortest length. Functional equations also arise frequently in theoretical physics (the Euler-Lagrange equations) and statistical finance. However, the techniques used to solve functional equations are typically very different than those used to solve algebraic equations (solving for variables) and require slightly more creativity. Through this packet, you'll learn a few techniques to get started solving simple functional equations.

The interesting thing about functional equations is that there isn't just one way to solve the problems. I'll guide you through some examples to build your intuition, but really you can always just try playing with the equations and see what you learn about the function!

2 Substitution

One technique for solving functional equations is substitution. Functional equations give us some facts about a function, so sometimes we can extract more information about the function by making a clever substitution. One example of this was in the warm-up, but let's see a slightly more challenging example; suppose we want to find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying $f(x) + 3f\left(\frac{1}{x}\right) = x^2$. This looks like a hard problem, but we can use substitution to make it easier. Let's try substituting $x \leftarrow \frac{1}{x}$. We do this because we have both an x and a $\frac{1}{x}$ in the equation, so we can try to exploit some symmetry:

$$f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x^2}.$$

But using this along with our original equations, we have a system of two equations in two unknowns:

$$\begin{cases} f(x) + 3f\left(\frac{1}{x}\right) = x^2 \\ f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x^2}. \end{cases}$$

We can make this clearer by setting $A = f(x)$ and $B = f\left(\frac{1}{x}\right)$:

$$\begin{cases} A + 3B = x^2 \\ 3A + B = \frac{1}{x^2}. \end{cases}$$

The second equation implies $B = \frac{1}{x^2} - 3A$, so we can substitute this into the first equation to get:

$$A + 3\left(\frac{1}{x^2} - 3A\right) = x^2 \implies -8A = x^2 - \frac{3}{x^2} \implies A = \frac{3}{8x^2} - \frac{x^2}{8}.$$

But $A = f(x)$, so this means:

$$f(x) = \frac{3}{8x^2} - \frac{x^2}{8}.$$

Therefore, we've solved the functional equation! With our example out of the way, let's try some problems in a similar flavor. These are ordered approximately in order of difficulty.

Problem 1. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x)^2 = 7x^2 + 8$.

Problem 2. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x)^2 = 7x^2 - 8$.

Problem 3. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(3x+5) = 7x^2 - 8$.

Problem 4. *Find all functions* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *satisfying* $f(x) + xf\left(\frac{1}{x}\right) = 3x$. *Here,* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *means that* f *takes in a nonzero number and gives back a real number. Hint: follow the pattern from the example and remember that you can divide by* x *(since we can't input 0 to* f *anyways).*

Problem 5. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x) + 3f(1-x) = 5x + 7$*. Hint: follow the pattern from the example (why does this work here?).*

Problem 6. *Find all functions f satisfying $5f(x) + 4f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = 7x^2$. Hint: follow the pattern from the example but substitute twice this time.*

3 Symmetry

When solving functional equations in multiple variables, we can often exploit symmetry to simplify the problem. For example, suppose we want to find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) - f(x-y) = f(x)f(y)$. The symmetry in the right-hand side of the equation between x and y suggests that substituting $x \leftarrow y$ and $y \leftarrow x$ is a good idea; we find:

$$f(x+y) - f(y-x) = f(x)f(y).$$

Using this equation and the original equation, we know that $f(x-y) = f(y-x)$. Substituting $y = 0$, this means that $f(x) = f(-x)$ for all x. Then, we can try substituting $y \leftarrow -y$ in our original equation to exploit our new knowledge to find (since $f(y) = f(-y)$):

$$f(x-y) - f(x+y) = f(x)f(-y) = f(x)f(y).$$

But using our original equation, this means that for all choices of x and y:

$$f(x)f(y) = f(x-y) - f(x+y) = -(f(x+y) - f(x-y)) = -f(x)f(y).$$

But the above equation forces $f(x) = 0$ for all x (why?) and so f is the zero function. Therefore, we've solved the functional equation! Let's try some problems in a similar flavor. These are also ordered approximately in order of difficulty.

Problem 7. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) + f(y) = 2f(y)$ for all choices of real numbers x and y. Hint: is there a symmetry here?*

Problem 8. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x, y) = x + 3 \cdot f(y, x)$. *Here,* $f(x, y)$ *means a function which takes in two numbers (*x *and* y*) and gives you back a number. Hint: try a substitution that exploits a symmetry in the equation, like we did in the example.*

Problem 9. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x - y) = f(x) + f(y) - 2xy$. *Hint: just plug in values to this equation and look for symmetries.*

Problem 10. *Find all functions f which satisfy $f(x^2+y) = f(x^3+2y) + f(x^4)$. Hint: try substituting a value of y which makes $x^2+y = x^3+2y$. This shows $f(x) = 0$ when $x \geq 0$ (why?). Then, what can you say about $f(x)$ when $x < 0$?*

Problem 11 (a). *The point of Problem 8 is to find all functions* $f : \mathbb{Q} \to \mathbb{Q}$ *which take in a fraction and give back a fraction (not just any real number) satisfying* $f(1) = 1$ *and* $f(xy) = f(x)f(y) - f(x+y) + 1$*. What is the value of* $f(2)$*,* $f(3)$*, and* $f(4)$*? Now describe what* f *does to positive whole numbers.*

Problem 11 (b). *Put* $y \leftarrow n$ *for a positive whole number* n *and* $x \leftarrow \frac{m}{n}$ *for a fraction* $\frac{m}{n}$ *to find the value of* $f\left(\frac{m}{n}\right)$*. If you're stuck, try Section 4 first.*

4 Reduction

This is a more advanced topic which highlights an interesting application of functional equations. If f is a function which takes in a fraction and gives back a fraction (not just any real number), we denote it $f : \mathbb{Q} \to \mathbb{Q}$. It's a well-known fact that all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying **Cauchy's functional equation** $f(x + y) = f(x) + f(y)$ are linear functions $f(x) = f(1) \cdot x$. This next section will guide you through showing this fact:

Problem 12 (a). *Show that if f satisfies Cauchy's functional equation, then $f(0) = 0$ and $f(1) = 1$.*

Problem 12 (b). *Compute $f(2)$, $f(3)$ and $f(4)$ if f satisfies Cauchy's functional equation. Do you see a pattern? Describe how to calculate $f(x)$ for any positive whole number x.*

Problem 12 (c). *Compute* $f\left(\frac{1}{2}\right)$, $f\left(\frac{1}{3}\right)$ *and* $f\left(\frac{2}{3}\right)$ *if* f *satisfies Cauchy's functional equation. Describe how to calculate* $f\left(\frac{m}{n}\right)$ *for any positive fraction* $\frac{m}{n}$.

Problem 12 (d). *Suppose* f *satisfies Cauchy's functional equation and assume we know* $f(x)$. *Describe how to find* $f(-x)$. *Can you use this to write down a solution to Cauchy's functional equation when* $f : \mathbb{Q} \to \mathbb{Q}$?

Okay, good. Now, we're armed with the solution to the Cauchy functional equation and we can put it to use. The way we use the Cauchy equation to solve other functional equations is to show that if a function satisfies the new equation, it must also satisfy the Cauchy equation. But we know every function satisfying the Cauchy equation is linear, so we've solved our new equation too! This is the idea of **reduction** to the Cauchy equation; let's put this idea in practice with a problem.

Problem 13 (Jensen's functional equation). *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying Jensen's equation $f(x) + f(y) = 2 \cdot f\left(\frac{x+y}{2}\right)$. Hint: make a substitution to show that if f satisfies Jensen's equation, it must also satisfy Cauchy's equation. Sidebar: this result means that the only functions which are both "convex" and "concave" ("curvy up" and "curvy down" respectively) are linear.*

Problem 14. *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(f(x)^2 + f(y)) = xf(x) + y$. Hint: show $f(f(y)) = y$; we say that f is an "involution". In particular, this means that f is its own inverse, so set $x = f(t)$ and $y = f(u)$ for some t and u (why is this allowed?). Now show that f satisfies the Cauchy functional equation.*

5 Challenge Problems

Problem 15. *If* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $f(x) + f\left(\frac{x}{1-x}\right) = 5$ *then find* $f(\pi)$.

Problem 16. *Find all "continuous" functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* $f(x) = f(x^2)$. *Here, "continuous" means that if you have a sequence of numbers* $(x_1, x_2, \cdots)$ *converging to* x *then the sequence* $(f(x_1), f(x_2), \cdots)$ *converges to* $f(x)$. *Don't worry if you don't know exactly what this means; just try plugging in some numbers and see what happens.*