Electrical Circuits and Random Walks

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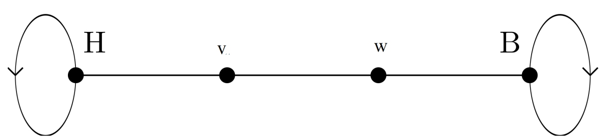
This worksheet is largely about graphs. As a reminder, a *graph* G consists of a set V_G of *vertices* of G, and a set E_G of edges connecting two vertices of G. For most of this worksheet, you can assume that G has no loops, that is, no vertex is connected to itself, and at most one edge connects any two vertices.

1 Random Walks

Consider a finite graph G. At one vertex, H, is home, at another vertex, B, is an all-consuming black hole. You find yourself at vertex x. If every minute you move along a randomly-selected edge from your current position to a neighboring vertex, what's the chance you successfully make it home before you stumble into the black hole?

1.1 A First Example

Consider a simple case, 4 vertices in a line, with H at one end, and B on the other. From either x or y, you pick one of the two directions to walk in, each with probability $\frac{1}{2}$, and once you have reached H or B you stay put.



Problem 1. For a given vertex x, $p(x)$ be the probability that if you start at x and walk randomly, you end up at H before B.

- What are $p(H)$ and $p(B)$?
- Given $p(w)$ and $p(H)$, find an expression for $p(v)$. Given $p(v)$ and $p(B)$, find an expression for $p(w)$.
- Use the last two parts to solve for $p(v)$ and $p(w)$.

*adapted by Aaron Anderson from notes by Padraic Bartlett and "Random walks and electric networks" by Doyle and Snell, with edits by Glenn Sun et al.

Solution.

- $p(H) = 1$ and $p(B) = 0$.
- $p(v) = \frac{1}{2}(p(H) + p(w))$ and $p(w) = \frac{1}{2}(p(v) + p(B))$.
- $p(v) = \frac{2}{3}$ and $p(w) = \frac{1}{3}$.

Problem 2. In a general graph G, if the vertex x, which is neither H nor B, has neighbors $y_1, y_2, \ldots, y_n$, what expression can we find for $p(x)$ in terms of $p(y_1), p(y_2), \ldots, p(y_n)$? How can we use this to solve for p in general?

Solution. We have $p(x) = \frac{1}{n}(p(y_1) + \cdots + p(y_n))$. If we write an equation for each vertex other than H and B, we have $|V| - 2$ linear equations in $|V| - 2$ variables, which we can solve.

Problem 3.

- Let G be the graph with the corners of a square as vertices, and the sides of the square as edges. If H and B are opposite corners of the square, find $p(x)$ for all vertices x.
- Instead of the corners and edges of a square, let G consist of the corners and edges of a 3-d cube (with H and B now opposite on the cube). Find p again.

Solution.

- $p(x) = \frac{1}{2}$ for the two nontrivial vertices.
- $p(x) = \frac{3}{5}$ for the three corners adjacent to H and $p(x) = \frac{2}{5}$ for the three corners adjacent to B.

1.2 Weighted Edges

So far, we've assumed that these random walks happen uniformly randomly, that is, whenever you choose a move randomly, each of the possible moves has the same probability. Now consider the case where you have a preference between edges to walk on, so while you still make a random choice, not all choices are equally likely. Specifically, we give each edge xy a *weight*, w_{xy}. This'll just be a nonnegative real number.

We keep $p(H) = 1$ and $p(B) = 0$ as before, but for other vertices x, we change the probability of choosing an edge to be proportional to its weight. That is, if you are currently at vertex x, then you will move to a neighboring vertex y with probability

$$\mathbb{P}[\text{move to } y \text{ from } x] = \frac{w_{xy}}{w_x},$$

where w_x is the total of all the weights of all the edges at x; that is,

$$w_x = \sum_{y \text{ is a neighbor of } x} w_{xy}.$$

Problem 4. Explain why we can assume each w_{xy} is a *positive* real number.

Solution. If some $w_{xy} = 0$, it is equivalent to just delete the edge between x and y from the graph.

Problem 5. Similarly to Problem 2, derive a new expression for $p(x)$ in terms of the values $p(y_1), \dots, p(y_n)$ and $w_{xy_1}, \dots, w_{xy_n}$ where $y_1, \dots, y_n$ are the neighbors of x?

Solution.

$$p(x) = \frac{w_{xy_1} p(y_1) + \cdots + w_{xy_n} p(y_n)}{w_{xy_1} + \cdots + w_{xy_n}}$$

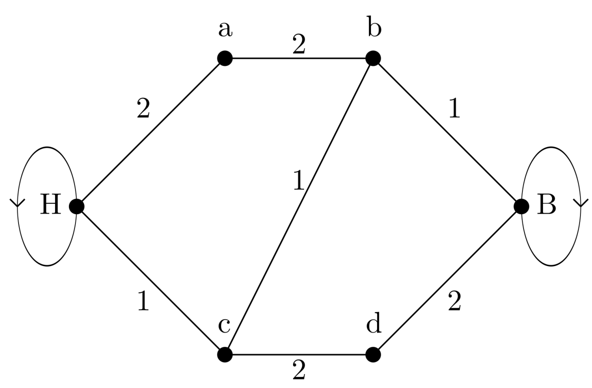
Problem 6. Let G be a graph with 5 vertices, H, B, x, y, z. Each pair in x, y, z is connected with an edge, so these three vertices form a triangle, and H is connected only to x, while B is connected only to z. All edges have weight 1, except the edge xz, which has weight $w_{xz} = 2$. Find $p(x), p(y), p(z)$.

Solution. We write

$$p(x) = \frac{1 \cdot p(H) + 1 \cdot p(y) + 2 \cdot p(z)}{1 + 1 + 2}$$
$$p(y) = \frac{1 \cdot p(x) + 1 \cdot p(z)}{1 + 1}$$
$$p(z) = \frac{1 \cdot p(B) + 1 \cdot p(y) + 2 \cdot p(x)}{1 + 1 + 2}$$

Solving the system gives $x = \frac{7}{12}, y = \frac{1}{2}, z = \frac{5}{12}$.

Problem 7. Let's try a more complicated example.



- What expressions can we find for $p(a), p(b), p(c)$, and $p(d)$?

- Solve your system of equations. (Hint: If you want easier calculations, use symmetry.)[1]

Solution.

- We can write

$$p(a) = \frac{2 \cdot p(H) + 2 \cdot p(b)}{2+2}$$
$$p(b) = \frac{2 \cdot p(a) + 1 \cdot p(c) + 1 \cdot p(B)}{2+1+1}$$
$$p(c) = \frac{1 \cdot p(H) + 1 \cdot p(b) + 2 \cdot p(d)}{1+1+2}$$
$$p(d) = \frac{2 \cdot p(c) + 2 \cdot p(B)}{2+2}$$

- Solving the system gives $a = \frac{3}{4}, b = \frac{1}{2}, c = \frac{1}{2}, d = \frac{1}{4}$.

2 Circuits

And now for something completely different.

If you've worked with circuits with resistors before, or calculated things about them in a physics class, you'll be pretty familiar with this section. If not, don't worry, we will explain everything you need for this lesson.

An electrical circuit can be modeled as a graph G, subject to a few extra properties. We'll just be concerned with *voltage*, *current*, and *resistance*.

- Voltage is a function on the vertices - that is, it assigns each vertex x of G a number, $V(x)$, called the voltage at x. In any circuit we think about here, we pick a vertex to call the ground, and standardize $V(\text{ground}) = 0$. We pick another vertex to be a source of electricity, and (by attaching one end of a 1 volt battery to the ground, and another end to the source) we let $V(\text{source}) = 1$.
- Current is another function, but on the edges. Since current flows in a certain direction, we think of "oriented edges," which is the same as ordered pairs (x, y) where x and y are neighbors in the graph G. We'll use I_{xy} to quantify the current on the edge from x to y that is flowing from x to y - the current in the opposite direction had better be the opposite, so $I_{xy} = -I_{yx}$.
- Resistance is another function on the edges, but this time it doesn't depend on any direction, so we'll give any (unordered) edge $\{x, y\}$ a positive number R_{xy} to describe the resistance of $\{x, y\}$.

For our purposes, there will be only two rules, which correspond to actual physical laws. We encountered these a few weeks ago, but we'll restate them in our current notation.

- Ohm's Law: The current along the edge from x to y, I_{xy}, is given by

$$I_{xy} = \frac{V(x) - V(y)}{R_{xy}}$$

[1] Hint 2: The symmetry tells you that $p(a) = 1 - p(d)$ and $p(b) = 1 - p(c)$. Can you say why these are true by symmetry?

where $V(x)$ and $V(y)$ are the voltages of these points, and R_{xy} is the resistance along the edge $\{x, y\}$ (independent of direction).

- Kirchoff's Law: The sum of all currents pointing into a vertex is 0. If $x \in V_G$, and $N(x)$ is the set of neighbors of x,

$$\sum_{y \in N(x)} I_{xy} = 0$$

  This is because current is (basically) a flow of electrons, and the number of electrons flowing into a vertex is the same as the number flowing out, so the total change is 0.

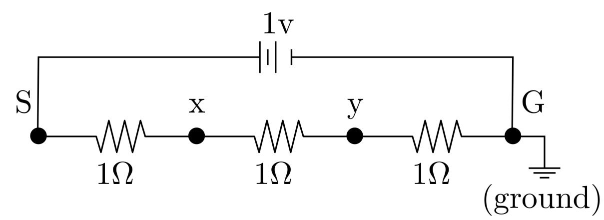
  EXCEPTION: This does not need to hold at the source or the ground vertices, where the extra current can go into or come out of the battery.

It will be convenient to also define *conductance*, which is just the reciprocal of resistance, so that the conductance C_{xy} of an edge is just $1/R_{xy}$. Then we also define (this you won't see in a physics class) the conductance $C_x = \sum_{y \in N(x)} C_{xy}$ for each vertex x to be the total conductance of the edges leading to it.

2.1 Calculating Voltage

Say we have a graph, with a particular vertex G that we set to be the ground, with $V(G) = 0$, and a vertex S which we set to be the source, with $V(S) = 1$. We also know the resistance R_{xy} of each edge $\{x, y\}$, and we will come up with a method for calculating the voltage $V(x)$ at every vertex x.

2.2 Example



This is the same graph from Figure 1, except this time as a circuit! (Note that the 1v battery is not actually an edge). We've turned it into a circuit by

- Replacing all the edges with resistors of resistance 1, measured in Ohms, hence the Ω (Capital Ohm-ega)
- Attaching a 1v battery between G and S.

Problem 8. We know by the setup of the problem that $V(G) = 0$ and $V(S) = 1$.

- **Using Kirchoff's Law:** What equations does Kirchoff's Law give us pertaining to currents out of x and currents out of y?
- **Using Ohm's Law:** Try using Ohm's Law to turn the equations about currents into equations about voltages and resistances, and find an expression for $V(x)$ in terms of other voltages, and an expression for $V(y)$ in terms of other voltages. Then solve the system of equations. Do these look familiar?

Solution.

- $I_{xS} + I_{xy} = 0$ and $I_{yx} + I_{yG} = 0$.
- Because R is always 1, this directly translates into voltages as $V(x) - V(S) + V(x) - V(y) = 0$ and $V(y) - V(x) + V(y) - V(G) = 0$, i.e. $V(x) = \frac{V(S)+V(y)}{2}$ and $V(y) = \frac{V(x)+V(G}{2}$. These are the same equations as in Problem 1, and the solutions is $V(x) = \frac{2}{3}$, $V(y) = \frac{1}{3}$.

3 The Equivalence

In the last problem, you should have found that the equations for $V(x)$ and $V(y)$ were the same as the equations for $p(x)$ and $p(y)$ in the random walk problem on the same graph. In general, we can translate from a graph random walk problem to an electrical circuit problem by translating the "black hole" vertex to the ground and the "home" vertex to the source. We will show that the probability function p solving the random walk problem with home a and black hole b equals the voltage function V of the circuit with source a and ground b.

Problem 9. What is the correct electrical quantity that corresponds to the weight of each edge? Give heuristic arguments, no need to be formal. (Hint: resistance or conductance, which is it, and why?)

Solution. The weights w_{xy} should be conductances C_{xy}. High weights means a random walk is more likely to choose that path, where as high resistance means electrons are less likely to choose that path, so it should be the opposite of resistance.

Now, we will formalize the equivalence by proving that p and V satisfy the same equations.

Problem 10. We saw earlier that the probability function p satisfies

$$p(x) = \sum_{y \text{ is a neighbor of } x} p(y) \frac{w_{xy}}{w_x}$$

for every vertex x other than a and b. Show that the voltage function V satisfies

$$V(x) = \sum_{y \text{ is a neighbor of } x} V(y) \frac{C_{xy}}{C_x},$$

using Ohm's Law and Kirchoff's Law.

Solution. Replicate the solution from Problem 8 in general. Start with Kirchoff's law to write

$$\sum_{y \in N(x)} I_{xy} = 0$$

for all x. Replace $I_{xy} = \frac{V(x)-V(y)}{R_{xy}}$ by Ohm's law. Pull out the $V(x)$ terms to get

$$V(x) \sum_{y \in N(x)} \frac{1}{R_{xy}} - \sum_{y \in N(x)} \frac{V(y)}{R_{xy}} = 0.$$

Rearranging and replacing $\frac{1}{R_{xy}}$ with C_{xy} and $\sum_{y \in N(x)} C_{xy} = C_x$ gives the desired

$$V(x) = \sum_{y \in N(x)} V(y) \frac{C_{xy}}{C_x}.$$

Thus if $w_{xy} = C_{xy}$, then the random walk probability function p and the voltage function V satisfy the same linear equations. To show that $p = V$, it suffices to show that there can only be one solution to these equations. We looked at some special cases of this earlier, but we leave the details here as an exercise.

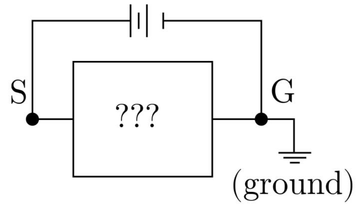
Problem 11. (Bonus) Prove that the system of equations we're studying has a unique solution. (Hint: We also need the fact that $p(H) = 1$ and $p(B) = 0$, or equivalently that $V(S) = 1$ and $V(G) = 0$.

Solution. Suppose p, q were two solutions. Then $p - q$ must also be a solution because the equations are linear, but $(p-q)(H) = (p-q)(B) = 0$, so all the weighted averages are just zero (several different ways to prove this, but it's intuitively obvious), so $p - q = 0$.

4 Effective Resistance

The most efficient way of calculating electrical properties of a circuit is actually not just setting up an equation for each vertex and solving the system. Instead, we try to simplify the problem, modifying the graph in a way that doesn't change our final answer.

Consider a graph interpreted as an electrical circuit, with a source vertex and a ground vertex. Imagine that the entire circuit between these two vertices (except for the battery and the ground) is stuffed into an obfuscating box, so that all we see is the battery attached to two vertices, each attached to this box containing an unknown circuit:



What information do we know about this box? We can still measure the current flowing out of the source, $\sum_{x \text{ is a neighbor of } S} I_{Sx}$, or the amount of current flowing into the ground, $\sum_{x \text{ is a neighbor of } G} I_{xG}$. Physically, these quantities should be equal, as they both describe the amount of current flowing into the battery on one side and out the other, and if they are not equal, charge would be building up in the battery or being depleted over time. However, we did not assume that currents coming in and out of the battery are equal in our graph-theoretic model, so we should still check that this is true.

Problem 12. Using Kirchoff's Law (which we only assumed holds at vertices other than S and G), prove that

$$\sum_{x \text{ is a neighbor of } S} I_{Sx} = \sum_{x \text{ is a neighbor of } G} I_{xG}$$

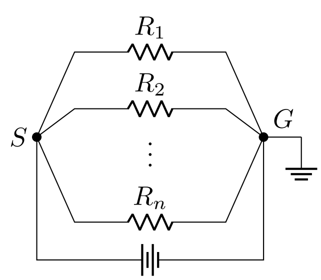
.

Solution. Add Kirchoff's law for all vertices $x \neq S, G$ to get$\sum_{x \neq S,G} \sum_{y \in N(x)} I_{xy} = 0$. This sum counts both I_{st} and I_{ts} for all edges $\{s, t\}$, except I_{st} when s is S or G. Because $I_{st} + I_{ts} = 0$, these cancel out, leaving us with just $\sum_{x \in N(S)} I_{Sx} + \sum_{x \in N(G)} I_{Gx} = 0$. Because $I_{Gx} = -I_{xG}$, we get the desired $\sum_{x \in N(S)} I_{Sx} = \sum_{x \in N(G)} I_{xG}$.

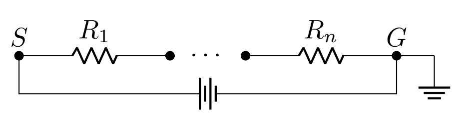
If we call this current $I_S = \sum_{x \in N(S)} I_{Sx}$, we can now pretend that the box contains just one big resistor, with current I_A flowing through it. This gives us enough information to calculate its resistance. Ohm's Law tells us that its resistance would have to be given by $\frac{V(S)-V(G)}{I_S}$, which we will call R_{eff}, the *effective* resistance.

Problem 13. Using Ohm's Law and Kirchoff's Law, calculate the effective resistance R_{eff} of these two circuits:

- Resistors in *parallel* (Note: this is the first time we're seeing multiple edges connecting the same two vertices. All of our formulas basically still work though, do you see what you have to do?)



- Resistors in *series*



Solution.

- Anyway, suppose the power source provides V volts. Denote I_i the current across resistor R_i pointing to the right. By Ohm's law, $I_i = \frac{V}{R_i}$. Then the source current is $I_S = \sum_{i=1}^n I_i = V(\frac{1}{R_1} + \cdots + \frac{1}{R_n})$. Then the effective resistance is

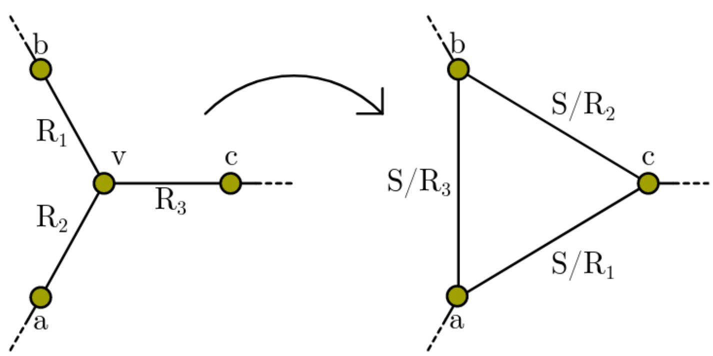
$$R_{\text{eff}} = \frac{V}{I_S} = \frac{1}{\frac{1}{R_1} + \cdots + \frac{1}{R_n}}.$$

- With the same notation, by Kirchoff's law all of the currents are the same, i.e. $I_i = \cdots = I_n$. Call this I for simplicity, the source current I_S is also I. Let V_i denote the voltage at the node to the left of R_i. We have by Ohm's law, $V_i - V_{i+1} = IR_i$. Taking the sum over all i, the sum telescopes, and we get $V(S) - V(G) = I(R_1 + \cdots + R_n)$. Dividing gives

$$R_{\text{eff}} = R_1 + \cdots + R_n.$$

Now that we know how to calculate the effective resistance of certain circuits, we can use these effective resistances to simplify other complicated circuits. Whenever we have resistors in parallel or series, that is, a portion of the circuit resembling one of these diagrams, we can replace this portion with a single resistor, with resistance determined by the effective resistance of the piece of circuit it's replacing.

Problem 14. ($\Delta - Y$ Transform) Consider the following two circuits:



Show that if $S = R_1R_2 + R_1R_3 + R_2R_3$, the effective resistance between any two of a, b, c are the same for both circuits. (It is often useful to turn one of these circuits into the other!)

Solution.

Without loss of generality, we check a and b. The effective resistance in the left diagram is $R_1 + R_2$, and in the right diagram is, using equations for parallel and series resistors:

$$\frac{S}{R_3} \parallel \left(\frac{S}{R_1} + \frac{S}{R_2}\right) = \frac{S}{R_3} \parallel \frac{S(R_1 + R_2)}{R_1R_2} = \frac{1}{\frac{R_3}{S} + \frac{R_1R_2}{S(R_1+R_2)}} = \frac{1}{\frac{R_3R_1+R_3R_2+R_1R_2}{S(R_1+R_2)}} = R_1 + R_2.$$

Problem 15. For each of the Problems 1, 6, and 7, redraw the graphs as circuits and find the effective resistances between S and G.

Problem 16. Suppose that we make a circuit by connecting the 2^n vertices of an n-dimensional cube with 1Ω resistors at the edges of the cube. If we place the ground and source at opposite vertices of the cube, what is the effective resistance of this circuit? (You can leave the answer as a summation.)