

ORMC Olympiad Group  
Winter: Week 8  
Analysis: Functions and Polynomials

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## Problems

1. (**Refail Alizade**) Let  $P(x)$  be a third degree polynomial whose all roots are real.  $P$  also satisfies the condition that whenever  $P(t) = 0$  then  $P(t + 1) = 1$ . If 7 is a root of  $P(x)$  find all possible polynomials.
2. (**NAMO 2010**)  $p \neq 0$  and  $a, b, c$  are complex roots of the polynomial  $x^3 + px + 1 = 0$ . Define

$$A = \frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1}$$

What is  $p \cdot A$ ?

3.  $a, b, c \in \mathbb{R}$  and the polynomial  $p(x) = x^4 + ax^2 + bx + c$  have one root  $P(3 + 4i) = 0$ . If  $P(x) \geq 0$  for all  $x$ , find least possible value of  $c$ .
4. Let  $(x-1)(x-3)(x-5) \cdots (x-49) = x^{25} + a_{24}x^{24} + a_{23}x^{23} + \cdots + a_1x + a_0$ . What is the remainder when  $a_{23}$  is divided by 1000?
5. (**PSS P6**) Prove  $a, b, c \in \mathbb{R}, a + b + c > 0, bc + ca + ab > 0, abc > 0 \Rightarrow a, b, c > 0$ .

6. **(TNMO-FR 2013)** Find the sum of real roots of the following equation

$$x^4 - 8x^3 + 13x^2 - 24x + 9 = 0$$

**HINT: Try to factorize. It can be factorized as  $(x^2 - ax + b)(x^2 - cx + d)$ . You need to find appropriate real values  $a, b, c, d$ ,**

7. **PSS E8** Find the real solutions of the equation  $\sqrt[4]{x} + \sqrt[4]{97 - x} = 5$
8. **(IMO 2004)** Find all polynomials  $f$  with real coefficients such that for all reals  $a, b, c$  such that  $ab + bc + ca = 0$  we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

9. **(IMO 2019)** Let  $\mathbb{Z}$  be the set of integers. Determine all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a$  and  $b$ ,

$$f(2a) + 2f(b) = f(f(a + b)).$$

10. **(PSS P11)** Solve the equation  $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0$ .
11. **(Turkey NMO 2021)** If a polynomial with real coefficients of degree  $d$  has at least  $d$  coefficients equal to 1 and has  $d$  real roots, what is the maximum possible value of  $d$ ?
12. **(PSS P37)** Find all polynomial solutions of the functional equation  $f(x)f(x + 1) = f(x^2 + x + 1)$ .