ORMC AMC 10/12 Group
More Inequalities

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1 Warmup: AM-GM Review

1. Find the minimum value of $x + \frac{1}{x}$ when x is a positive real number.

2. Find the minimum value of $\sqrt{x + \frac{1}{y}} + \sqrt{y + \frac{1}{x}}$ when $x, y > 0$.

3. Find the minimum value of $\frac{x^2+2}{\sqrt{x^2+1}}$, over all real numbers x.

4. Find the minimum value of $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$, where x, y, z are positive real numbers.

5. Suppose that $x + y = 1$. Find the minimum value of $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)$.

2 Cauchy-Schwarz Inequality

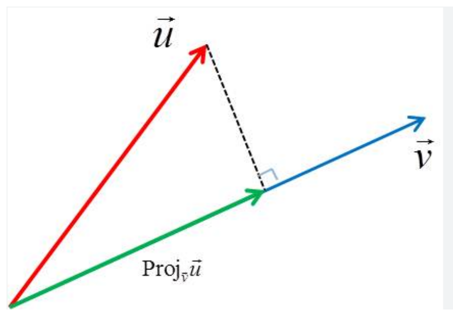
The Cauchy-Schwarz Inequality traditionally involves vectors, but it can be viewed slightly more generally in terms of two sets of real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. The statement of the inequality is:

$$\boxed{(a_1^2 + a_2^2 + a_3^3 + \cdots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n)^2}$$

For those of you who are familiar with vectors, we can view the a_i's as components of a vector $u = (a_1, a_2, \ldots, a_n)$, and the b_i's as components of a vector $v = (b_1, b_2, \ldots, b_n)$. Then, in terms of vectors, what the Cauchy-Schwarz inequality says is that $\boxed{\|u\|\|v\| \geq |u \cdot v|}$, where $\|v\|$ is the norm (magnitude/length) of a vector v, and $u \cdot v$ is the usual dot product:

$$u \cdot v = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n.$$

When thinking in terms of vectors, the inequality becomes more obvious. The dot product $u \cdot v$ is equivalent to $\|Proj_v(u)\|\|w\|$, where $Proj_v(u)$ is the component of u parallel to v, or the projection of u onto v



In the figure above, we can see that there is a right triangle in which u forms the hypotenuse, and $Proj_v(u)$ forms one of the legs. So, this gives us $|u \cdot v| = \|Proj_v(u)\|\|v\| \leq \|u\|\|v\|$. In particular, as some of you may already know, $|u \cdot v| = \|u\|\|v\| \cos(\theta)$, where θ is the angle between $uandv$.

We can show the explicit form involving the a_i's and b_i's directly, using AM-GM. Expanding the left hand side, we get:

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^3 + \cdots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2) &= \sum_{i,j\in[[1,n]]} a_i^2b_j^2 = \sum_{i=j\in[[1,n]]} a_i^2b_j^2 + \sum_{i\neq j\in[[1,n]]} a_i^2b_j^2 \\ &= \sum_{i\in[[1,n]]} a_i^2b_i^2 + \sum_{1<i<j<n} \left(a_i^2b_j^2 + a_j^2b_i^2\right)\end{aligned}$$

Expanding the right hand side gives:

$$\begin{aligned}(a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n)^2 &= \sum_{i,j\in[[1,n]]} a_ib_ia_jb_j = \sum_{i=j\in[[1,n]]} a_ib_ia_jb_j + \sum_{i\neq j\in[[1,n]]} a_ib_ia_jb_j \\ &= \sum_{i\in[[1,n]]} a_i^2b_i^2 + \sum_{1<i<j<n} (a_ib_ia_jb_j + a_jb_ja_ib_i) = \sum_{i\in[[1,n]]} a_i^2b_i^2 + \sum_{1<i<j<n} (2a_ib_ia_jb_j)\end{aligned}$$

Notice that the first summation is the same in both expansions, so we can focus only on the second summation. Apply AM-GM to each term of the second summation from the LHS:

$$a_i^2b_j^2 + a_j^2b_i^2 \geq 2\sqrt{(a_i^2b_j^2)(a_j^2b_i^2)} = 2a_ib_ja_jb_i = 2a_ib_ia_jb_j.$$

So, every term in the LHS is $\geq$ the corresponding term on the RHS, which means that $LHS \geq RHS$. In particular, the inequality follows:

$$\boxed{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.}$$

2.1 Examples

1. Show that

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$$

2. Let x, y, z be positive real numbers. Find the minimum value of

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

2.2 Exercises

1. Let a, b, c be positive real numbers. Prove that

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

2. Jane has drawn 6 rectangles, each with a different length and width. The sum of the squares of the lengths is 40, and the sum of the squares of the widths is 20. What is the largest possible sum of the areas of the rectangles?

3. Let x, y be positive real numbers. Find the maximum value of

$$\frac{(3x+4y)^2}{x^2+y^2}.$$

4. If a, b, c, d are positive, show that

$$2\sqrt{a+b+c+d} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

5. If a, b, c, d are positive real numbers, show that

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

3 Mean Inequality Chain

We begin by defining two new means:

1. The **quadratic mean** (QM) of $a_1, a_2, \ldots, a_n$ is the value

$$\boxed{\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.}$$

   This is sometimes called the *root mean square*, since it is the square root of the average of the squared values.

2. The **harmonic mean** (HM) of nonzero real numbers $a_1, a_2, \ldots, a_n$ is the value

$$\boxed{\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.}$$

Together with AM and GM, we get the *mean inequality chain* QM-AM-GM-HM:

$$\boxed{\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.}$$

It is unlikely that you run into a situation where you need all 4 of these at once, but if any 2 of them come up, it is good to know that they are all ordered, and what order they come in. It is also important to remember that this mean inequality chain should only be applied to positive real numbers.

The fact that $QM \geq AM$ follows from Cauchy-Schwarz. We define $b_1 = b_2 = b_3 = \cdots = b_n = 1$, and we get:

$$(n \cdot QM)^2 = n^2 \cdot \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}^2 = n(a_1^2 + a_2^2 + \cdots + a_n^2)$$
$$= (1^2 + 1^2 + \cdots + 1^2)(a_1^2 + a_2^2 + \cdots + a_n^2) = (b_1^2 + b_2^2 + \cdots + b_n^2)(a_1^2 + a_2^2 + \cdots + a_n^2)$$
$$\geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 = (a_1 + a_2 + \cdots + a_n)^2 = (n \cdot AM)^2.$$

So, it follows that $QM \geq AM$.

The GM-HM inequality can actually be proved directly from AM-GM! The first thing to notice is how n is suspiciously in the numerator of the HM expression. In order to put it in a form that looks more like QM and AM, we can consider its reciprocal:

$$1/(HM) = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\frac{1}{a_2}\cdots\frac{1}{a_n}} = \sqrt[n]{\frac{1}{a_1 a_2 a_3 \cdots a_n}} = \frac{1}{\sqrt[n]{a_1 a_2 a_3 \cdots a_n}} = 1/(GM)$$

So, using AM-GM, we have shown that $1/(HM) \geq 1/(GM)$, which then implies that $GM \geq HM$.

The inequality chain $QM \geq AM \geq GM \geq HM$ follows.

3.1 Examples

1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers which satisfy $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq n^2.$$

2. Prove that $\sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} \geq \sqrt{ab} - \frac{2}{\frac{1}{a}+\frac{1}{b}}$.

3.2 Exercises

1. If x, y, z are all positive, find the minimum possible value of

$$\left(1+\frac{x}{2y}\right)\left(1+\frac{y}{2z}\right)\left(1+\frac{z}{2x}\right).$$

2. If a, b, c are positive, show that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

3. If a, b, c are positive, show that

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

4. Let $a, b > 0$. Show that

$$\frac{a+b}{2}-\sqrt{ab} \geq \sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}.$$

5. Let m, n be (not necessarily distinct) positive integers. Find the minimum value of $x^m + \frac{1}{x^n}$.