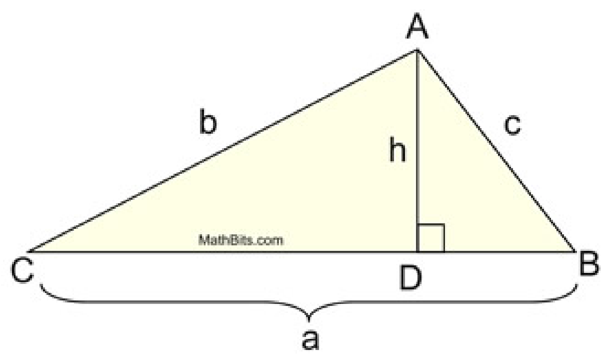
ORMC AMC 10/12 Group
Sequences & Series

January 29, 2023

1 Warmup: Trig Review

Review of some formulas that we discussed previously:

In the triangle below, we can calculate the area as $\frac{1}{2}a \cdot h$, or we can use trigonometry to only involve side lengths and angles, instead of referencing the height directly:



$$h = b \cdot \sin(C) \implies K = \boxed{\frac{1}{2}a \cdot b\sin(C)}$$

So, for any two sides of a triangle (x, y), and the angle between them (Z), the area is $\frac{1}{2}xy\sin(Z)$. We can write this out for each pair of sides in the diagram above:

$$K = \frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B)$$

If we multiply every one of these expressions by $\frac{2}{abc}$, we get the (extended) ***Law of Sines***:

$$\frac{2K}{abc} = \boxed{\frac{\sin(C)}{c} = \frac{\sin(A)}{a} = \frac{\sin(B)}{b}}.$$

Another thing we can try with this triangle diagram is applying the pythagorean theorem, since h creates two right triangles. We have:

$$b^2 = (CD)^2 + h^2, \quad c^2 = (a - CD)^2 + h^2 = a^2 - 2a(CD) + (CD)^2 + h^2$$

Subtracting the first equation from the second to get eliminate $(CD)^2 + h^2$, we get the ***Law of Cosines***:

$$c^2 - b^2 = a^2 - 2a(CD) = a^2 - 2ab\cos(C) \implies \boxed{c^2 = a^2 + b^2 - 2ab\cos(C)}.$$

1.1 Exercises

1. In $\triangle ABC$, we have $AB = 13$, $BC = 14$, and $AC = 15$. Point P lies on BC, and $AP \perp BC$. What is the length of BP?

2. In $\triangle ABC$, $\angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.

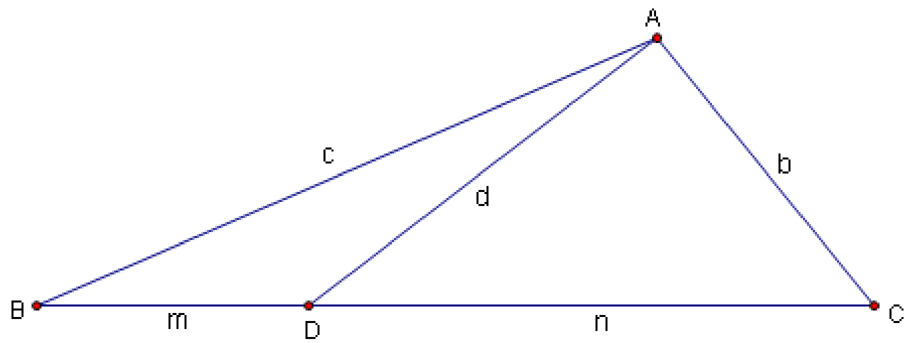
3. **(2012 AMC 12A #16)** Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

4. **(2017 AMC 12B #15)** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

5. **(2022 AMC 10B #20)** Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

2 Stewart's Theorem

Stewart's Theorem is a useful statement about the relationship between side lengths of a triangle and the length of of an arbitrary segment from one vertex of the triangle to an opposite side (such a segment is called a *cevian*). Consider the following diagram:



Letting $a = m + n$, Stewart's theorem states that these lengths satisfy:

$$\boxed{b^2m + c^2n = amn + d^2a}.$$

The power of Stewart's theorem is that it allows us to compute side lengths in an arbitrary triangle, with an arbitrary cevian, without involving any of the angles!

We can prove this using Law of Cosines on the triangles created by segment $\overline{AD}$. Recall that $\cos(x) = -\cos(180° - x)$. With this in mind, we want to use angles $\angle ADC$ and $\angle BDA$, so that the cosines will cancel out and leave us with just side lengths. For $\triangle BDA$ we have:

$$m^2 + d^2 - c^2 = 2md\cos(\angle BDA) \implies \cos(\angle BDA) = \frac{m^2 + d^2 - c^2}{2md},$$

and for $\triangle CDA$ we have:

$$n^2 + d^2 - b^2 = 2nd\cos(\angle CDA) \implies \cos(\angle CDA) = \frac{n^2 + d^2 - b^2}{2nd},$$

$$\implies \cos(\angle BDA) = -\frac{n^2 + d^2 - b^2}{2nd}$$

$$\implies \frac{m^2 + d^2 - c^2}{2md} = -\frac{n^2 + d^2 - b^2}{2nd} \implies \frac{m^2 + d^2 - c^2}{m} = -\frac{n^2 + d^2 - b^2}{n}$$

$$\implies m^2n + d^2n - c^2n = -n^2m - d^2m + b^2m$$

$$\implies (m^2n + n^2m) + (d^2n + d^2m) = c^2n + b^2m \implies \boxed{mna + d^2a = c^2n + b^2m}.$$

2.1 Examples

1. Two of the sides of a triangle have length 4 and 9. The median drawn to the third side of this triangle has a length of 6. Compute the length of the third side of this triangle.

2. **(2022 AMC 10A #23)** Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

2.2 Exercises

1. In $\triangle ABC$, X is a point on $\overline{AB}$ such that $AX = 2, BX = 4$. If $AC = 7$ and $BC = 5$, compute CX.

2. **(2013 AMC 12A # 19)** In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

3. **(2002 AMC 12B # 23)** In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?

4. **(2003 AIME I #7)** Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of $\triangle ACD$. Find s.

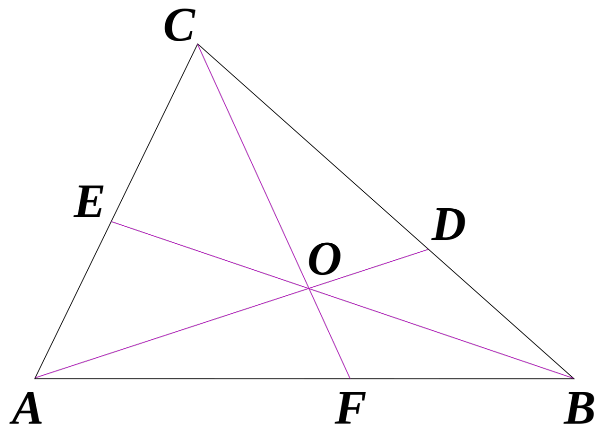
5. **(2013 AIME II #13)** In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

3 Mass Points

Mass Points is a technique used to solve for lengths in triangles (in general, polygons). The more general form of mass points is called *barycentric coordinates*, which some of you may be interested in, but is beyond the scope of the AMC 10/12.

Mass points involves systematically assigning "weights" to points in a diagram, using ratios of lengths. Specifically, when some point C divides a line segment $\overline{AB}$, A and B may be assigned *masses* W_A and W_B, respectively. We want $\overline{AB}$ to be able to "balance" on point C, so the weights and segments would satisfy the equation: $\frac{W_A}{W_B} = \frac{BC}{AC} \iff (AC) \cdot W_A = (BC) \cdot W_B$.

Consider the following diagram:



The way to systematically assign weights is as follows:

1. First, choose a central point for the entire figure to "balance" around. Usually, you will have a diagram with one or more cevians that intersect in 1 point, and you want this point of intersection to be the balancing point. In the diagram above, this point would be O.

2. You can assign a weight to any point in the diagram, without loss of generality

3. Deduce the appropriate weights for other points, following a few simple rules:
   - Any cevian should balance on the central point (O).
   - Any side of the triangle should balance on the point where a cevian intersects it.
   - If a segment $\overline{AB}$ is "balanced" on some point C, then the weights W_A, W_B should satisfy the equation from above: $(AC) \cdot W_A = (BC) \cdot W_B$.
   - If C is the balancing point of $\overline{AB}$, then $W_A + W_B = W_C$.

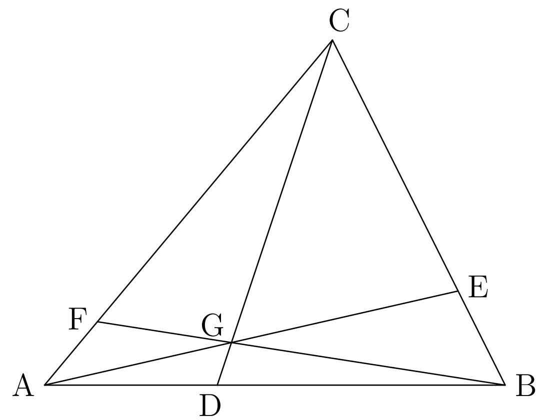
The key limitation of mass points is that any "balancing point" is unique. So, each time we use mass points, we can only have 1 central point where all the cevians intersect, and we can only consider 1 balancing point for each side of the triangle at a time.

The technique is best illustrated with some examples– see the next page.

3.1 Examples

1. $\triangle ABC$ has point D on AB, point E on BC, and point F on AC. AE, CD, and BF intersect at point G. The ratio $AD : DB$ is $3 : 5$ and the ratio $CE : EB$ is $8 : 3$. Find the ratio of $FG : GB$

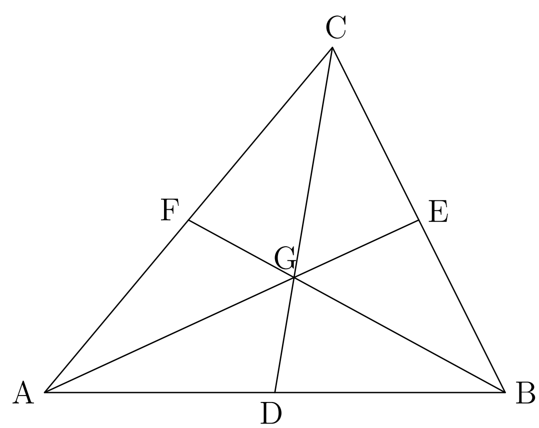
   Solution:

   By mass points, $FG : GB = W_B : W_F$. We may begin WLOG by assigning B a weight of 1, since the segments whose ratios we know are AB and CB. This means that $W_A = \frac{5}{3}W_B = \frac{5}{3}$ and $W_C = \frac{3}{8}W_B = \frac{3}{8}$. Since F balances A and C, we then have $W_F = \frac{5}{3} + \frac{3}{8} = \frac{40+9}{24}$. This gives us $\frac{W_B}{W_F} = \frac{1}{\frac{49}{24}} = \boxed{\frac{24}{49}}$.

2. Consider a triangle ABC with its three medians drawn, with the intersection points being D, E, F, corresponding to AB, BC, and AC respectively. Let the centroid be G. What are the ratios $AG : GE, BG : GF, CG : GD$?

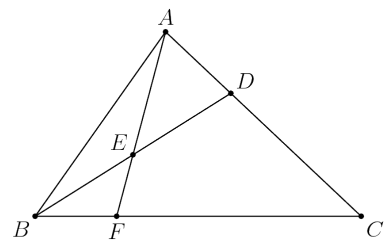
   Solution:

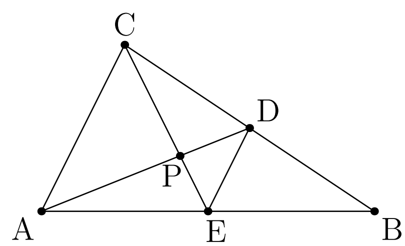
By the definition of a median, we know that D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{BC}$, and F is the midpoint of $\overline{AC}$. In particular, this means that $W_A = W_B = W_C$, and $W_D = W_E = W_F = 2W_A$. So, $W_E = 2W_A, W_F = 2W_B$, and $W_D = 2W_C$. So, all of the ratios are $\boxed{2 : 1}$.

3.2 Exercises

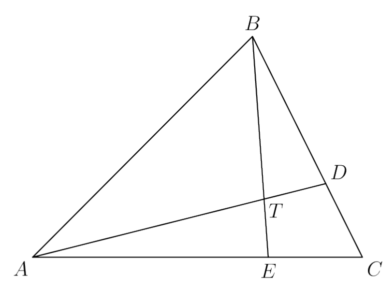
1. **(2019 AMC 8 #24)** In triangle $\triangle ABC$, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line $\overline{BC}$ and line $\overline{AE}$. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



2. **(2013 AMC 10B #16)** In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

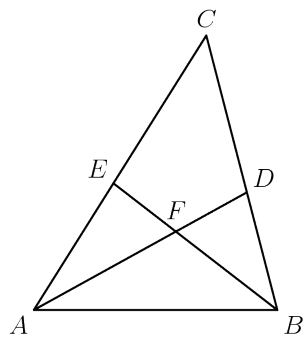


3. **(2004 AMC 10B #20)** In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?

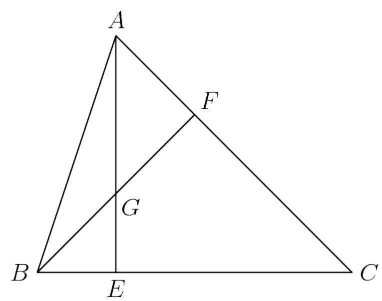


4. **(2016 AMC 10A #19)** In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of $r, s,$ and t is 1. What is $r + s + t$?

5. **(2016 AMC 12A #12)**In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



6. **(1971 AHSME #26)** In $\triangle ABC$, point F divides side AC in the ratio $1 : 2$. Let E be the point of intersection of side BC and AG where G is the midpoints of BF. What is $BE : EC$?



7. **(2009 AIME I #5)** Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

8. **(2011 AIME II #4)** In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.