ORMC AMC 10/12 Group
Roots of Unity and AM-GM Inequality

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1 Roots of Unity

Let n be a natural number. The n-th roots of unity are the complex numbers x such that $x^n = 1$. Let $\omega = e^{i2\pi/n}$. Using Euler's formula, it can be shown that the n-th roots of unity are w^k for $k = 0, 1, \ldots, n-1$. It is not hard to see that these n points lie evenly spaced on the unit circle.

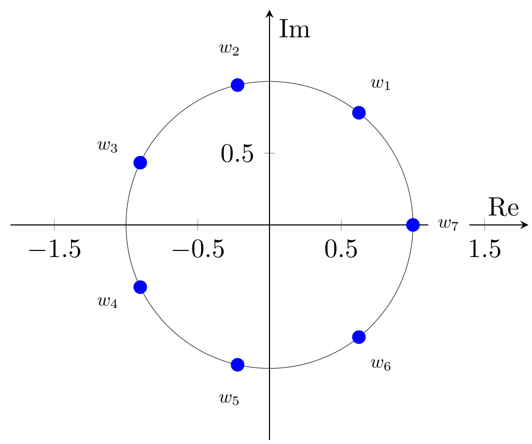

Figure 1: 7^{th} roots of unity on $\mathbb{C}$-plane

Proof: Fix $n \in \mathbb{N}$, and let $w = e^{i2\pi/n}$. We will show that the roots of the equation $x^n = 1$ are $1, w, w^2, \ldots, w^{n-1}$. For all $k \in \{0, 1, \ldots, n-1\}$, we have that

$$\begin{aligned}(w^k)^n &= ((e^{i2\pi/n})^k)^n \\ &= ((e^{i2\pi/n})^n)^k \\ &= (e^{i2\pi})^k \\ &= 1.\end{aligned}$$

Thus, for all $k \in \{0, 1, \ldots, n-1\}$, w^k is a solution to $x^n = 1$, and collectively they are the n-th roots of unity.

1.1 Examples

1. An equilateral triangle has its centroid located at the origin and a vertex at $(-1, 0)$. What are the coordinates of the other two vertices?

2. There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

1.2 Exercises

1. **(2017 AMC 12B #12)** What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

2. **(2012 AIME I #6)** The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin \frac{m\pi}{n}$, for relatively prime positive integers m and n with $m < n$. Find n.

3. **(2018 AIME I #6)** Let N be the number of complex numbers z with the property that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

4. **(2021 AMC 12A #22)** Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}$, and $\cos \frac{6\pi}{7}$, where angles are in radians. What is abc?

2 AM-GM

The AM-GM inequality is an important inequality that is frequently found on the AMC. Most often, you will see the two-variable case:

$$\frac{x+y}{2} \geq \sqrt{xy}, \quad \forall\, x, y \geq 0$$

The inequality above states that the arithmetic mean (AM) of two nonnegative variables is greater than or equal to the geometric mean of those variables. This comes from the "trivial inequality", which states that $x^2 \geq 0$, $\forall x \in \mathbb{R}$. In particular, we can "work backwards" towards the trivial inequality:

$$\frac{x+y}{2} \geq \sqrt{xy} \iff x+y \geq 2\sqrt{xy} \iff x+y-2\sqrt{xy} \geq 0$$

This is where it becomes important that $x, y \geq 0$, so that $x = \sqrt{x}^2, y = \sqrt{y}^2$. Hopefully you recognize the form above as a squared binomial:

$$x+y-2\sqrt{xy} = (\sqrt{x}-\sqrt{y})^2$$

Clearly this value is nonnegative (by the trivial inequality), so the AM-GM inequality follows.

It is also important to know the general case, for n variables:

$$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

We can show this using a special type of induction called Cauchy Induction which involves showing that:

1. If the statement is true for n, then it is also true for $2n$.
2. If the statement is true for $n+1$, then it is also true for n.

Proof. As the base case for our induction, we use the $n = 2$ case that we already proved above.

1. Suppose the statement is true for some $k \in \mathbb{N}$. Then, following the proof of the base case:

$$\frac{x_1+\cdots+x_k}{k} \geq \sqrt[k]{x_1 \cdots x_k}, \qquad \frac{x_{k+1}+\cdots+x_{2k}}{k} \geq \sqrt[k]{x_{k+1} \cdots x_{2k}}$$

for any nonnegative $x_1, \cdots, x_{2k}$. By the $n = 2$ case,

$$\frac{x_1+\cdots+x_{2k}}{2k} = \frac{\frac{x_1+\cdots+x_k}{k} + \frac{x_{k+1}+\cdots+x_{2k}}{k}}{2} \geq \sqrt{\sqrt[k]{x_1 \cdots x_k}\,\sqrt[k]{x_{k+1} \cdots x_{2k}}} = \sqrt[2k]{x_1 \cdots x_{2k}}.$$

So if the statement is true for $n = k$, then it is also true for $n = 2k$.

2. The second part involves showing that $n-1$ is actually a special case of n. Suppose that the statement is true for some $k > 1$. Then, consider the values: $x_1, x_2, \ldots, x_{k-1}, x_k$, where $x_k = \sqrt[k-1]{x_1 x_2 \cdots x_{k-1}}$. We have:

$$\frac{x_1+\cdots+x_{k-1}+x_k}{k} \geq \sqrt[k]{x_1 x_2 \cdots x_{k-1} x_k} = x_k^{(k-1)/k}\sqrt[k]{x_k} = x_k$$

$$\implies \frac{x_1+\cdots+x_{k-1}}{k} \geq x_k - \frac{1}{k}x_k = \frac{k-1}{k}x_k \implies \frac{x_1+\cdots+x_{k-1}}{k-1} \geq \sqrt[k-1]{x_1 x_2 \cdots x_{k-1}}.$$

So, the statement is true for all n, by induction. □

Important note: For AM-GM, equality is achieved if and only if all of the x_i's are equal.

For those of you who don't know calculus, AM-GM tends to be a good way to solve "find the maximum/minimum" type problems. And for those of you who do know calculus, AM-GM can be a quicker way to find a maximum or minimum (compared to taking a derivative, solving, then plugging back in) where it is applicable.

2.1 Examples

1. Find the maximum of $2 - a - \frac{1}{2a}$ for all positive a.

2. Let $a > 5$. Find the smallest possible value of $a + \frac{4}{a-5}$.

2.2 Exercises

1. Suppose that $(a + 1)(b + 1)(c + 1) = 8$ and $a, b, c \geq 0$. Show that $abc \leq 1$.

2. **(1983 AIME #9)** Find the minimum value of $\frac{9x^2 \sin^2 x+4}{x \sin x}$ for $0 < x < \pi$.

3. **(2000 AMC 12 #12)** Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C$?

4. A ball is thrown upward from the top of a tower. If its height at time t is described by $-t^2 + 60t + 700$, what is the greatest height the ball reaches?

5. **(2020 AMC 12B #22)** What is the maximum value of $\frac{(2^t-3t)t}{4^t}$ for real values of t?

6. **(USAMTS 31/1/3)** Circle ω is inscribed in unit square $PLUM$, and points I and E lie on ω such that U, I, and E are collinear. Find, the greatest possible area for $\triangle PIE$

7. (Hard) If the polynomial $x^3 - 12x^2 + ax - 64$ has only real, nonnegative roots, find a.