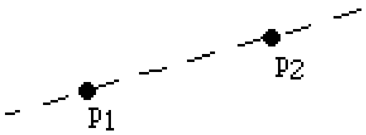
Origami

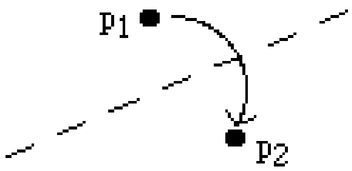
Instructors

Here are the axioms of origami:

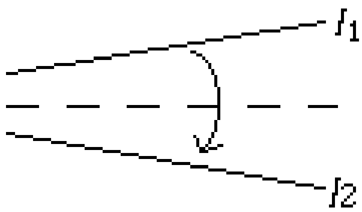
(O1) Given two points p_1 and p_2 we can fold a line connecting them.



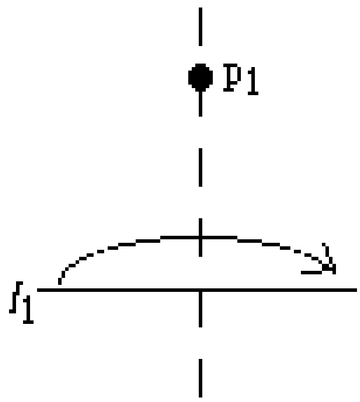
(O2) Given two points p_1 and p_2 we can fold p_1 onto p_2.



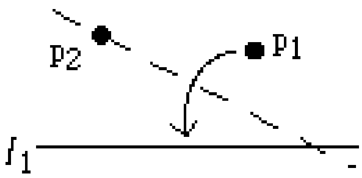
(O3) Given two lines l_1 and l_2 we can fold line l_1 onto l_2.



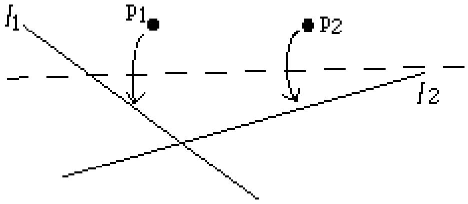
(O4) Given a point p_1 and a line l_1 we can make a fold perpendicular to l_1 passing through the point p_1.



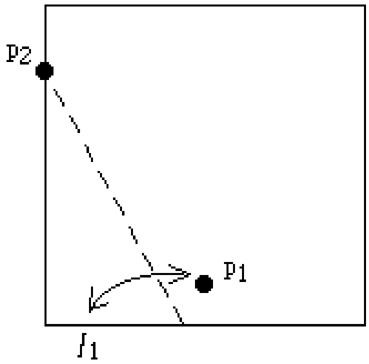
(O5) Given two points p_1 and p_2 and a line l_1 we can make a fold that places p_1 onto l_1 and passes through the point p_2.



(O6)[Beloch fold] Given two points p_1 and p_2 and two lines l_1 and l_2 we can make a fold that places p_1 onto line l_1 and places p_2 onto line l_2.



Problem 1.
(a) Take a piece of paper. Let the bottom edge be l_1 and take p_1 to be a point in the middle and close to l_1. Then choose p_2 to be anywhere on the left or right edge of the square and perform axiom (O5). Then choose a different p_2. Repeat this 8 or 9 times keeping the same p_1 and choosing different p_2's. What do you see?



(b) On another piece of paper draw random two intersecting lines l_1 and l_2 and points p_1 and p_2 about an inch close to their intersection. Perform a Beloch fold for them.

Problem 2.
(a) Given a circle, its center, and a point p on the circle, use origami to construct a tangent line to the circle that passes through p.

(b) Given a circle, its center, and a point p on the circle, use origami to construct an equilateral triangle inscribed in the circle that passes through p.

(c) Given a triangle, use origami to construct the center of the circle inscribed in it and its tangent points.

Problem 3.
(a) Emulate axiom O5 with a compass and straightedge.

(b) In your emulation, probably, there is a choice which of the two intersections of a circle and a line to take. Does it mean that there are two ways to perform the fold?

Problem 4.
(a) Construct a regular hexagon using a ruler and compass.

(b) Cut the triangle with angles 72°, 72°, 36° into 2 isosceles triangles.

(c) Using triangle similarity, prove that the ratio of the sides in this triangle is equal to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

(d) Find a way to construct a regular pentagon using only a ruler and compass.

Problem 5.
Prove that $\sqrt[3]{2} \neq \frac{a}{b}$ for any $a, b \in \mathbb{N}$.

Problem 6.
(a) Use origami to divide a given segment into 3 equal parts.

(b) Use origami to divide a given segment into n equal parts.

Problem 7.
(a) In the lecture, you saw that axioms (O1)-(O5) are all able to be simulated by compass and straightedge constructions. Is the following claim a correct *deduction* from the above? (In other words, does simulating (O1)-(O5) *prove* the claim?)

Claim: "In all cases, origami constructions are at least as powerful as compass and straightedge construction."

(b) Is the claim true? Argue *both* sides with yourself (or with a classmate).

(c) (Hard) Prove the sense in which the claim is true. (Hint: recall from lecture that all constructible lengths with straightedge and compass are rational, or of the form $a + b\sqrt{c}$ with a, b, c rational, or of the form $d + e\sqrt{f}$ with d, e, f of the form $a + b\sqrt{c}$ with a, b, c rational, etc.)

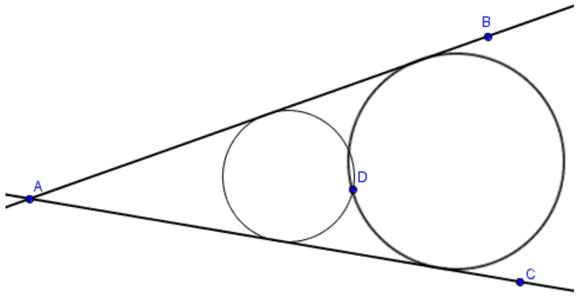
Problem 8.
Do each of the following with a compass and ruler.
Folds may be used, but are not required.

- **a:** Divide a circle into two parts of equal area.
- **b:** Divide a circle into three parts of equal area.
- **c:** Divide a circle into five parts of equal area.
- **d:** Divide a circle into seven parts of equal area.
- **e:** Divide a circle into n parts of equal area.

Problem 9.
Using a compass and ruler, find two circles tangent to a point D and lines AB and AC. (Problem of Appolonius, PLL case)

Hint 1: Construct the bisector.

Hint 2: All circles tangent to AB and AC are homothetic with centre at A. What does this mean?


B
A
D
C