AMC 8 Training: Combinatorics

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1 Warm Up

Let R be a set of nine distinct integers. Six of the elements are 2, 3, 4, 6, 9, and 14. What is the the number of possible values of R?

(A) 4 B) 5 C) 6 D) 7 E) 8

How many positive 4 digit integers have four different digits, wehre the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

(A) 24 (B) 48 (C) 60 (D) 84 (E) 108

A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76-game schedule. How many games does a team play within its own division?

(A) 36 (B) 48 (C) 54 (D) 60 (E) 72

2 Permutations and Combinations

2.1 Permutations

In a set of elements, a permutation is simply a rearrangement of its elements. The number of ways to re-order all n objects is simply $n! = 1 \times 2 \cdots \times n$. To re-order r objects of of the set of n objects when order matters, we can use a permutation:

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

This permutation formula often works best when all the objects in the set are different. If you have identical objects, you can switch their places and still have the same permutation. In this case, we need to make sure we do not over count by dividing by $n!$ for each repeated object.
If the i-th element has a_i copies, then the total number of permutations is:

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!a_2!\ldots a_n!}.$$

Examples:

1. Using only the letters given in each name, find the number of unique permutations of letters you can make.

a. BOYAN

b. CLAIRE

c. FATIMA

d. RONNIE

2. Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. Find the probability that the second term is 2.

2.2 Combinations

A combination is a way to choose a subset from a set of elements. In a combination, order does not matter. The number of ways to choose r out of n objects where order doesn't matter is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

We can also find this using Pascal's Triangle to iteratively find the *binomial coefficient*.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

3 Other Counting Strategies

3.1 Complementary Counting

In some cases, it's easier to count the elements that aren't needed in a set. It would be easier to count the number of elements and subtract those that you don't need. This is called *complementary counting*.

3.2 Stars and Bars

Stars and Bars is a strategy used to sort identical elements into distinct groups. It utilizes stars ($*$) and bars ($|$).

If we had 7 people who like chocolate ice cream and 3 people who have vanilla ice cream, we can sort them as such: $*******|***$

The divider bar $|$ represents the separation between the two groups, while the stars $*$ represent the elements in each group.

In general, the number of ways to split n identical objects split among m people is equivalent to chosing $m-1$ dividers among $n+(m-1)$ positions for a total of
$\binom{n+m-1}{m-1}$.

4 Practice

A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

(A) 9 (B) 18 (C) 27 (D) 36 (E) 77

A positive integer divisor of 12! is chosen at random. The probability hat the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 3 (B) 5 (C) 12 (D) 18 (E) 23

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 (B) 16 (C) 18 (D) 19 (E) 21

A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 (B) 288 (C) 312 (D) 1260 (E)40320

The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 33 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $\frac{1}{21}$ (B) $\frac{1}{4}$ (C) $\frac{5}{63}$ (D) $\frac{2}{21}$ (E) $\frac{1}{7}$

How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 (B) 6 (C) 12 (D) 18 (E) 24

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

(A) 105 (B)114 (C) 190 (D) 210 (E) 380

How many distinguishable arrangements are there of 1 brown tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

(A) 210 (B) 420 (C) 630 (D) 840 (E) 1050

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\frac{29}{128}$ (B) $\frac{23}{128}$ (C) $\frac{1}{4}$ (D) $\frac{35}{128}$ (E) $\frac{1}{2}$