Pidgeonhole Problems

Prepared by Mark on February 24, 2023

Problem 1:
Difficulty: Easy
Is it possible to cover an equilateral triangle with two smaller equilateral triangles? Why or why not?

Problem 2:
Difficulty: Hard
You are given $n + 1$ integers. Prove that there exist at least two of them such that their difference is divisible by n.

Problem 3:
Difficulty: Easy
You are given an 8×8 chess board with a pair of opposite corner squares cut off. You are further given a set of dominoes each equal in size to a pair of the board squares with a common side. Is it possible to tile the board with the dominoes in such a way that all the board squares are covered while the dominoes neither overlap nor stick out?

Problem 4:
Difficulty: Easy
The ocean covers more than a half of the Earth's surface. Prove that the ocean has at least one pair of antipodal points.

Problem 5:
Difficulty: Normal
There are $n > 1$ people at a party. Prove that among them there are at least two people who have the same number of acquaintances at the gathering. (We assume that if A knows B, then B also knows A)

Problem 6:
Difficulty: Normal
Show that among any five points with integer coordinates in the plane, there exist two such that the center of the line segment that connects them has integer coordinates as well.

Problem 7:
Difficulty: Normal
Prove that if every point on a straight line is painted either black or white, then there exist three points of the same color such that one is the midpoint of the line segment formed by the other two.

Problem 8:
Difficulty: Easy
All the points in the plane are painted with either one of two colors. Prove that there exist two points in the plane that have the same color and are located exactly one foot away from each other.

Problem 9:
Difficulty: Normal
Each point on a circle is colored either black or white. Prove that there exist three equally spaced points of the same color.

Problem 10:
Difficulty: Hard
Let n be an integer not divisible by 2 and 5. Show that n has a multiple consisting entirely of ones.

Problem 11:
Difficulty: Hard
Prove that for any $n > 1$, there exists an integer made of only sevens and zeros that is divisible by n.

Problem 12:
Difficulty: Hard
You choose $n + 1$ integers between 1 and $2n$. Show that you must select two co-prime numbers.

Problem 13:
Difficulty: Hard
You choose $n + 1$ integers between 1 and $2n$. Show you must select two numbers a and b such that a divides b.

Problem 14:
Difficulty: Hard
Prove that it is always possible to choose a subset of the set of integral numbers $a_1, a_2, ..., a_n$ so that the sum of the numbers in the subset is divisible by n.

Problem 15:
Difficulty: Hard
Prove that there exists a positive integer divisible by 2013 that has 2014 as its last for digits.

Problem 16:
Difficulty: Hard
Let n be an odd number. Let $a_1, a_2, ..., a_n$ be a permutation of the numbers $1, 2, ..., n$. Prove that the product $(a_1 - 1) \times (a_2 - 2) \times ... \times (a_n - n)$ is an even number.

Problem 17:
Difficulty: Hard
A stressed-out student consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days the student drinks exactly 100 espressos.

Problem 18:
Difficulty: Hard
Prove that at a party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Problem 19:
Difficulty: Brutal
Given a table with a marked point, O, and with 2013 properly working watches put down on the table, prove that there exists a moment in time when the sum of the distances from O to the watches' centers is less than the sum of the distances from O to the tips of the watches' minute hands.

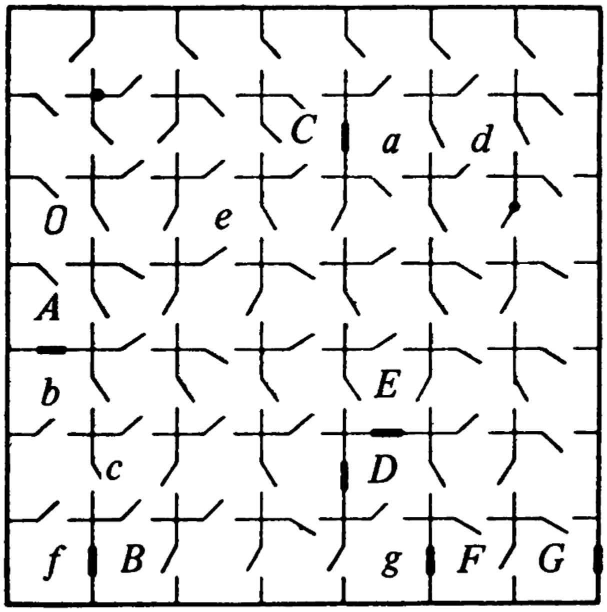
Bonus Problems

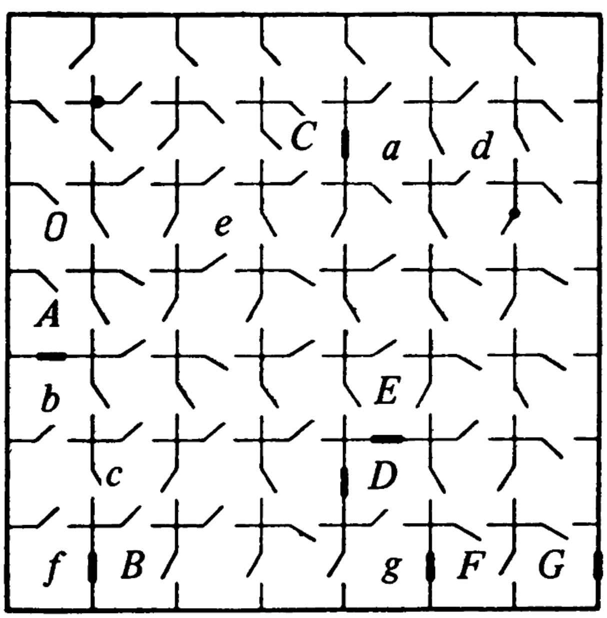
Problem 20:
There is one 1-by-4 battleship on a 10-by-10 field. What is the minimum number of shots you must take to find it?

Problem 21:
Can you perfectly tile a 10-by-10 grid using 1-by-4 rectangles?
Provide a proof.

Problem 22: How Does the Prisoner Escape?
This dungeon has 49 cells. In 7 cells (A to G in the diagram) there is a locked door (black bar). There are matching keys in cells a to g. All other doors open only from one side, as shown.

How does the prisoner in cell O escape through cell G?





Problem 23: One Hundred and Fourty-Five Doors

A prisoner was thrown into a medieval dungeon with 145 doors. Nine, shown by black bars, are locked, but each one will open if the prisoner passes through exactly 8 open doors before he reaches it. The prisoner doesn't need to go through every open door, but he must go through every cell and all 9 locked doors. If he enters a cell or goes through a door a second time, its doors will clang shut, trapping hum.

The prisoner started in the lower right corner, and escaped through the door in the top left. What was his route?