Part 1: Axioms of Origami

Axiom 1:
Given two points, we can fold a line between them.

Axiom 2:
Given two points, we can make a fold that places one atop the other.

Axiom 3:
Given two lines, we can make a fold that places one atop the other.

Axiom 4:
Given a point and a line, we can make a fold through the point and perpendicular to the line.

Axiom 5:
Given two points and a line, we can make a fold through one point that places the second on the line.

Axiom 6:
Given two points and two lines, we can make a fold that places each point on a line.
Problem 1:
Proposed by Nikita

a: Take a piece of paper. Let the bottom edge be $l_1$ and take $p_1$ to be a point in the middle and close to $l_1$. Then choose $p_2$ to be anywhere on the left or right edge of the square and perform Axiom 5. Then choose a different $p_2$. Repeat this 8 or 9 times keeping the same $p_1$ and choosing different $p_2$’s. What do you see?

b: Then, take another piece of paper. Draw two random intersecting lines $l_1$ and $l_2$ and points $p_1$ and $p_2$ about an inch close to their intersection. Perform a Beloch fold for them.
Problem 2:
Proposed by James

a: Given a circle, its center, and a point \( p \) on the circle, use origami to construct a tangent line to the circle that passes through \( p \).

b: Given a circle, its center, and a point \( p \) on the circle, use origami to construct an equilateral triangle inscribed in the circle that passes through \( p \).

c: Given a triangle, use origami to construct the center of the circle inscribed in it and its tangent points.

Problem 3:
Proposed by Nikita
Use origami to find the other three notable points in the given triangle: circumcenter, centroid and orthocenter.
Problem 4:
Proposed by Nikita

a: Emulate Axiom 5 with a compass and straightedge.
b: In your emulation, probably, there is a choice of which of the two intersections of a circle and a line to take. Does it mean that there are two ways to perform the fold?

Problem 5:
Proposed by Nikita

Prove that $\sqrt{2} \neq \frac{a}{b}$ for any $a, b \in \mathbb{N}$. 
Problem 6:
Proposed by Nikita

a: Construct a regular hexagon using a ruler and compass.
b: Cut the triangle with angles 72°, 72°, 36° into 2 isosceles triangles.
c: Using triangle similarity, prove that the ratio of the sides in this triangle is equal to the golden ratio \( \phi = \frac{1+\sqrt{5}}{2} \).
d: Find a way to construct a regular pentagon using only a ruler and compass.
Problem 7:
Proposed by Nikita

a: Use origami to divide a given segment into 3 equal parts.

b: Use origami to divide a given segment into \( n \) equal parts.

Problem 8:
Proposed by ?

a: In the lecture, you saw that Axioms 1–5 are all able to be simulated by compass and straightedge constructions. Is the following claim a correct *deduction* from the above? (In other words, does simulating Axioms 1–5 *prove* the claim?)

Claim: “In all cases, origami constructions are at least as powerful as compass and straightedge constructions.”

b: Is the claim true? Argue both sides with yourself (or with a classmate).

c: (Hard) Prove the sense in which the claim is true. (Hint: recall from the lecture that all constructible lengths with straightedge and compass are rational, or of the form \( a + b\sqrt{c} \) with \( a, b, c \) rational, or of the form \( d + e\sqrt{f} \) with \( d, e, f \) of the form \( a + b\sqrt{c} \) with \( a, b, c \) rational, etc.)
Problem 9:  
Proposed by Mark  
Do each of the following with a compass and ruler.  
Do not use folds.  

a: Divide a circle into five parts of equal area.  
b: Divide a circle into seven parts of equal area.  
c: Divide a circle into $n$ parts of equal area.
Problem 10:
Proposed by Sunny
Using a compass and ruler, find two circles tangent to a point D and lines AB and AC. (Problem of Appolonius, PLL case)

*Hint:* All circles tangent to $AB$ and $AC$ are homothetic with centre at $A$. What does this mean? Also, the angle bisector may help.