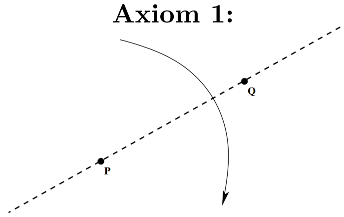
Origami Construction

Prepared by everyone on March 2, 2023

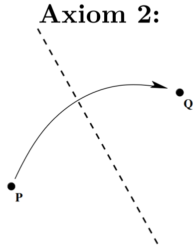
Part 1: Axioms of Origami

Axiom 1:



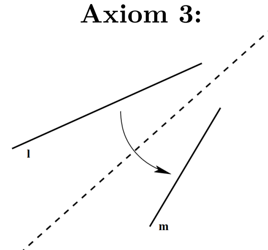
Given two points, we can fold a line between them.

Axiom 2:



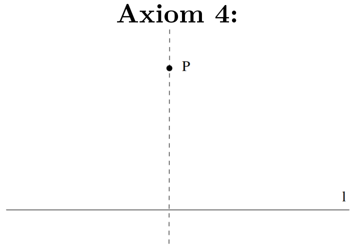
Given two points, we can make a fold that places one atop the other.

Axiom 3:



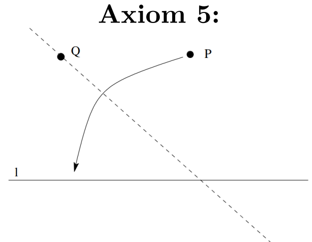
Given two lines, we can make a fold that places one atop the other

Axiom 4:



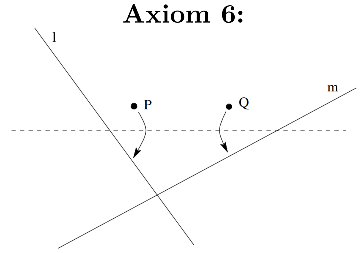
Given a point and a line, we can make a fold through the point and perpendicular to the line.

Axiom 5:



Given two points and a line, we can make a fold through one point that places the second on the line.

Axiom 6:

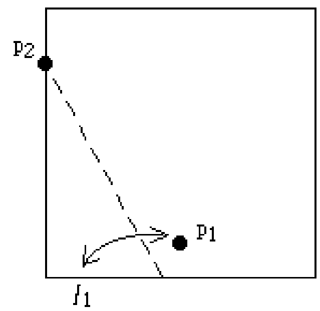


Given two points and two lines, we can make a fold that places each point on a line.

Problem 1:

Proposed by Nikita

a: Take a piece of paper. Let the bottom edge be l_1 and take p_1 to be a point in the middle and close to l_1. Then choose p_2 to be anywhere on the left or right edge of the square and perform Axiom 5. Then choose a different p_2. Repeat this 8 or 9 times keeping the same p_1 and choosing different p_2's. What do you see?



b: Then, take another piece of paper. Draw two random intersecting lines l_1 and l_2 and points p_1 and p_2 about an inch close to their intersection. Perform a Beloch fold for them.

Problem 2:

Proposed by James

- **a:** Given a circle, its center, and a point p on the circle, use origami to construct a tangent line to the circle that passes through p.
- **b:** Given a circle, its center, and a point p on the circle, use origami to construct an equilateral triangle inscribed in the circle that passes through p.
- **c:** Given a triangle, use origami to construct the center of the circle inscribed in it and its tangent points.

Problem 3:

Proposed by Nikita

Use origami to find the other three notable points in the given triangle: circumcenter, centroid and orthocenter.

Problem 4:
Proposed by Nikita

- **a:** Emulate Axiom 5 with a compass and straightedge.
- **b:** In your emulation, probably, there is a choice of which of the two intersections of a circle and a line to take. Does it mean that there are two ways to perform the fold?

Problem 5:
Proposed by Nikita
Prove that $\sqrt[3]{2} \neq \frac{a}{b}$ for any $a, b \in \mathbb{N}$.

Problem 6:

Proposed by Nikita

- **a:** Construct a regular hexagon using a ruler and compass.
- **b:** Cut the triangle with angles $72°$, $72°$, $36°$ into 2 isosceles triangles.
- **c:** Using triangle similarity, prove that the ratio of the sides in this triangle is equal to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.
- **d:** Find a way to construct a regular pentagon using only a ruler and compass.

Problem 7:

Proposed by Nikita

- **a:** Use origami to divide a given segment into 3 equal parts.
- **b:** Use origami to divide a given segment into n equal parts.

Problem 8:

Proposed by ?

- **a:** In the lecture, you saw that Axioms 1–5 are all able to be simulated by compass and straightedge constructions. Is the following claim a correct *deduction* from the above? (In other words, does simulating Axioms 1–5 *prove* the claim?)
  Claim: "In all cases, origami constructions are at least as powerful as compass and straightedge constructions."
- **b:** Is the claim true? Argue *both* sides with yourself (or with a classmate).
- **c:** (Hard) Prove the sense in which the claim is true. (Hint: recall from the lecture that all constructible lengths with straightedge and compass are rational, or of the form $a + b\sqrt{c}$ with a, b, c rational, or of the form $d + e\sqrt{f}$ with d, e, f of the form $a + b\sqrt{c}$ with a, b, c rational, etc.)

Problem 9:

Proposed by Mark

Do each of the following with a compass and ruler.
Do not use folds.

- **a:** Divide a circle into five parts of equal area.
- **b:** Divide a circle into seven parts of equal area.
- **c:** Divide a circle into n parts of equal area.

Problem 10:

Proposed by Sunny

Using a compass and ruler, find two circles tangent to a point D and lines AB and AC. (Problem of Appolonius, PLL case)

Hint: All circles tangent to AB and AC are homothetic with centre at A. What does this mean? Also, the angle bisector may help.