AMC 8 Training: Trigonometry

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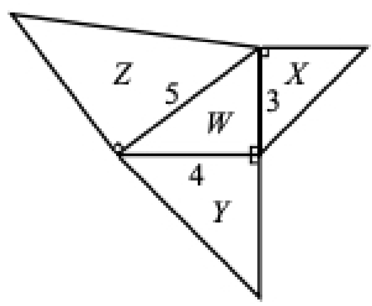
February 26, 2023

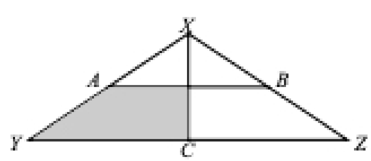
1 Warm Up

Right isosceles triangles are constructed on the sides of a 3-4-5 right triangle, as shown. A capital letter represents the area of each triangle. Which one of the following is true?



A) $X + Z = W + Y$ B) $W + X = Z$ C) $3X + 4Y = 5Z$
D) $X + W = 1/2(Y + Z)$ E) $X + Y = Z$

The area of triangle XYZ is 8 square inches. Points A and B are mid points of congruent segments XY and XZ. Altitude XC bisects YZ. What is the area (in square inches) of the shaded region?

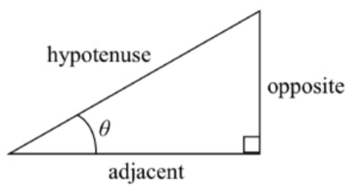


(A) 3/2 (B) 2 (C) 5/2 (D) 3 (E) 7/2

2 Introduction

Trigonometry is all about triangles! It can help us find angles and distances, which are useful in science and engineering. It can also show up on advanced math competitions, like the AMC.

In trigonometry, the most important triangle is a right triangle, which we learned about last week.



The right angle is denoted by the square, while another angle is labelled θ. The adjacent side is the "straight" edge that sits next to $\angle\theta$, while the opposite side is the edge that $\angle\theta$ opens into. The hypotenuse is the longest side.

From these three sides, we can use the three main trigonometric functions: $sin\theta$, $cos\theta$, and $tan\theta$

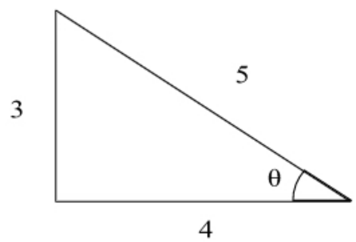
The three functions are as follows:

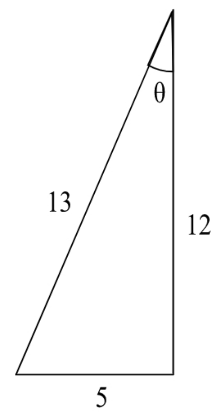
$$\sin(\theta) = \frac{opposite}{hypotenuse}, \quad \cos(\theta) = \frac{adjacent}{hypotenuse}, \quad \tan(\theta) = \frac{opposite}{adjacent}$$

This can easily be remembered with the phrase SOH-CAH-TOA.

2.1 Examples

Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the triangles below.

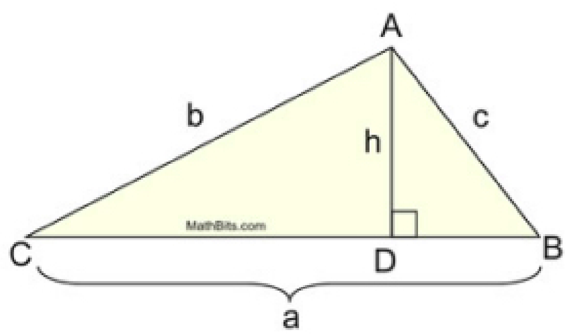




In rectangle $ABCD$, $\overline{AB} = 20$ and $\overline{BC} = 10$. Let E be a point on $\overline{CD}$ such that $\angle CBE = 15°$. What is $\overline{AE}$?

(A) $\frac{20\sqrt{3}}{3}$ (B) $10\sqrt{3}$ (C) 18 (D) $11\sqrt{3}$ (E) 20

Using the trig functions, can you find a formula for the area of the triangle below?

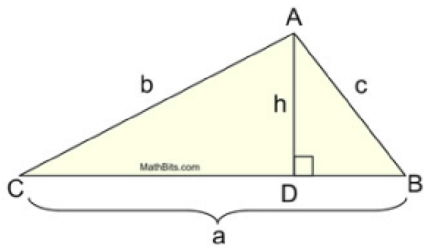


Now, use the above triangle to find $\sin B$ and $\sin D$.

3 Law of Sines and Cosines

3.1 Law of Sines

We can use the Law of Sines to solve for the angles and sides of a triangle.



Using sin$B$ and sin$C$ (which you solved in the last example question), we have the following equations:

$$\begin{array}{lcl} (1)\ \text{sinB} = \frac{h}{c} & \rightarrow & c\sin B = \text{h} \\ (2)\ \sin C = \frac{h}{b} & \rightarrow & b\sin C = \text{h} \end{array}$$

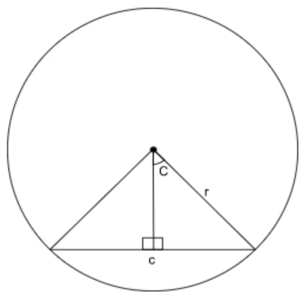
From here, we can set equation (1) and equation (2) equal:

$$c\sin B = b\sin C$$
$$\frac{b}{sinB} = \frac{c}{sinC}$$

We can follow similar steps to include $\frac{a}{sinA}$. Thus, we see that the Law of Sines is:

$$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$$

When a triangle is inscribed in a circle, we can use an extension of the Law of Sines.



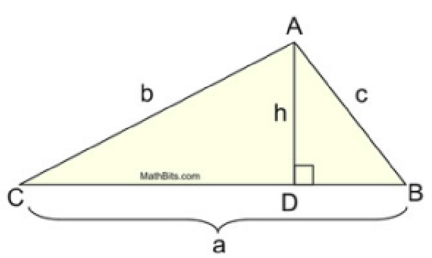
We can use $\sin\theta{=}\frac{opposite}{hypentuse}$ to note that:

$\sin C{=}\frac{\frac{c}{2}}{r}$
$\sin C{=}\frac{c}{2r}$
$2r\sin C{=}c$
$2r{=}\frac{c}{sinC}$

We can now relate this to the original Law of Sines:

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2r$$

3.2 Law of Cosines



Using the same $\triangle ABC$, we can find the Law of Cosines. Splitting it into $\triangle ACD$ and $\triangle ABD$, we can apply the Pythagorean Theorem to get two equations:

$b^2 = (CD)^2 + h^2$
$c^2 = (a - CD)^2 + h^2 = a^2 - 2a(CD) + (CD)^2 + h^2$

We can then subtract the first equation from the second to get the Law of Cosines:

$$c^2 - b^2 = a^2 - 2a(CD) = a^2 - 2ab\cos(C) \implies \boxed{c^2 = a^2 + b^2 - 2ab\cos(C)}.$$

4 Practice

In $\triangle ABC$, we have $AB = 13$, $BC = 14$, and $AC = 15$. Point P lies on BC, and AP bisects BC. What is the length of BP?

In $\triangle ABC$, we have $AB = 13$, $\angle A = 75°$, and $\angle B = 45°$. What are the perimeter and area of triangle ABC? (Hint: $\sin(75°) = \frac{\sqrt{2}+\sqrt{6}}{4}$)

In $\triangle ABC$, $\angle B = 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.

Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

(A) 9:1 (B) 16:1 (C) 25:1 (D) 36:1 (E) 37:1

In $\triangle ABC$ with integer side lengths, $\cos A = \frac{11}{16}$, $\cos B = \frac{7}{8}$, and $\cos C = -\frac{1}{4}$. What is the least possible perimeter for $\triangle ABC$?

(A) 9 (B) 12 (C) 23 (D) 27 (E) 44

In $\triangle ABC$, $\angle$A and $\angle$B measure 60°and 45°, respectively. The bisector of $\angle$A intersects BC at T, and $AT = 24$. The area of $\triangle ABC$ can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.