ORMC Olympiad Group
Winter: Week 5
Analysis: Functions and Polynomials

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Problems

1. (Turkish NMO 2004 Second Round P4) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$

2. (IMO 1977) Let $f(n)$ be a function defined on the set of all positive integers and with all its values in the same set. Prove that if $f(n + 1) > f(f(n))$

   for each positive integer $n$, then $f(n) = n$ for each $n$.

3. (Turkish NMO 2008 Second Round P4) $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ satisfy the given conditions

   a) $f(0, 0) = 1$, $f(0, 1) = 1$,

   b) $\forall k \not\in \{0, 1\}$ $f(0, k) = 0$ and

   c) $\forall n \geq 1$ and $k$, $f(n, k) = f(n - 1, k) + f(n - 1, k - 2n)$

   find the sum $\sum_{k=0}^{\binom{2009}{2}} f(2008, k)$
4. Let \( f(x) = x^2 - ax + 2020 \) where \( a \) is a real number. Find \( a \) if \( f(2020) = f(1048) \).

5. Let \( f(x) = x^3 - 4039x^2 + Nx + 1 \) where \( N \) is an integer. Find the remainder of \( N \) when divided by 1000 if \( f(2020) = f(2019) \).

6. Let \( f(x) = x^2 - 1 \) and \( g(x) = x - 1 \). Find the sum of integers \( n \) which does not satisfy \((f(g(n))) > g(n - 1)\)

7. (HMMT 2005 General 2) Find three real numbers \( a < b < c \) satisfying:

\[
\begin{align*}
    a + b + c &= \frac{21}{4} \\
    \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{21}{4} \\
    abc &= 1
\end{align*}
\]

8. The polynomial \( P \) satisfies \( P(1) = 3, P(3) = 7 \). Find the remainder of the polynomial when divided by \( x^2 - 4x + 3 \)

9. (TNMO-FR 1998) Find the number of primes \( p \), such that \( x^3 - 5x^2 - 22x + 56 \equiv 0 \pmod{p} \) has no three distinct integer roots in \([0, p)\)

10. Polynomial \( P(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0 \) satisfies \( P(n) = 2^n \) for \( n = -1000, -999, \ldots, 999, 1000 \). Compute the sum of positive even indexed terms, ie compute \( a_{2020} + a_{2018} + \cdots + a_2 \)

11. Find the monic polynomial with least degree which makes \((x - 1)(x^2 - 1)(x^3 - 1)Q(x) \geq 0\) for all \( x \in \mathbb{R} \)

12. Let \( Q(x) \) be the monic polynomial with least degree possible which makes \((x^3 - 5x^2 + x - 5)(x^2 - 7x + 6)Q(x) \geq 0\) for all \( x > 4 \). What is \( Q(10) \)?
13. (IMOMath Polynomials P1) A monic polynomial \( f(x) \) of fourth degree satisfies \( f(1) = 10, f(2) = 20 \) and \( f(3) = 30 \). Determine \( f(12) + f(-8) \).

14. **Polynomial division**
For the following choices of \( P(x) \) and \( D(x) \) do polynomial division and represent
\[
P(x) = D(x)Q(x) + R(x)
\]
where \( Q \) is quotient polynomial and \( R \) is remainder with \( \text{deg}(R) < \text{deg}(D) \).

(a) \( P(x) = x^4 + 10x + 1, \ D(x) = x^2 + x + 1 \)

(b) \( P(x) = x^5 - x^4 + x^3 - x^2 + x - 1, \ D(x) = x^2 + 1 \)

(c) \( P(x) = x^5 - x^4 + x^3 - x^2 + x - 1, \ D(x) = x^2 - 2 \)

15. The polynomial \( P \) satisfies \( P(-2) = 13, P(0) = 3 \) and \( P(2) = 9 \). Find the remainder of the polynomial when divided by \( x^3 - 4x \).

16. (AMC12 2001) The parabola with equation \( p(x) = ax^2 + bx + c \) and vertex \((h, k)\) is reflected about the line \( y = k \). This results in the parabola with equation \( q(x) = dx^2 + ex + f \). Which of the following equals \( a + b + c + d + e + f \)?

(A) \( 2b \)  (B) \( 2c \)  (C) \( 2a + 2b \)  (D) \( 2h \)  (E) \( 2k \)

17. (AIME 1983) What is the product of the real roots of the equation \( x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45} \)?

18. Let \( P(x) \) be third degree polynomial with \( P(1) = 2, P(2) = 4 \) and \( P(3) = 6 \). If \( P(x) \) gives remainder 56 when it is divided by \( x + 2 \), find the reminder when \( P(x + 1) \) divided by \( x - 4 \)

19. Solve the following system in reals:
\[
x/y + y/z + z/x = 3
\]
\[
y/z + z/y + x/z = 3
\]
\[
x + y + z = 3
\]
20. Solve the following system in reals:

\[ x - y + z = 6 \]
\[ x^2 + y^2 + z^2 = 14 \]
\[ x^3 - y^3 + z^3 = 36 \]