ORMC Olympiad Group
Winter: Week 5
Analysis: Functions and Polynomials

Osman Akar

February 1, 2023

Problems

1. **(Turkish NMO 2004 Second Round P4)** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$

2. **(IMO 1977)** Let $f(n)$ be a function defined on the set of all positive integers and with all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then $f(n) = n$ for each n.

3. **(Turkish NMO 2008 Second Round P4)** $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ satisfy the given conditions

a) $f(0,0) = 1$, $f(0,1) = 1$,

b) $\forall k \notin \{0,1\}$ $f(0,k) = 0$ and

c) $\forall n \geq 1$ and k , $f(n,k) = f(n-1,k) + f(n-1,k-2n)$

find the sum $\sum_{k=0}^{\binom{2009}{2}} f(2008,k)$

4. Let $f(x) = x^2 - ax + 2020$ where a is a real number. Find a if $f(2020) = f(1048)$.

5. Let $f(x) = x^3 - 4039x^2 + Nx + 1$ where N is an integer. Find the remainder of N when divided by 1000 if $f(2020) = f(2019)$.

6. Let $f(x) = x^2 - 1$ and $g(x) = x - 1$. Find the sum of integers n which does not satisfy
$$(f(g(n))) > g(n-1)$$

7. (HMMT 2005 General 2) Find three real numbers $a < b < c$ satisfying:
$$\begin{aligned} a + b + c &= \frac{21}{4} \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{21}{4} \\ abc &= 1 \end{aligned}$$

8. The polynomial P satisfies $P(1) = 3$, $P(3) = 7$. Find the remainder of the polynomial when divided by $x^2 - 4x + 3$

9. (TNMO-FR 1998) Find the number of primes p, such that $x^3 - 5x^2 - 22x + 56 \equiv 0 \ (mod\, p)$ has no three distinct integer roots in $[0, p)$

10. Polynomial $P(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0$ satisfies $P(n) = 2^n$ for $n = -1000, -999, \ldots, 999, 1000$. Compute the sum of positive even indexed terms, ie compute $a_{2020} + a_{2018} + \cdots + a_2$

11. Find the monic polynomial with least degree which makes
$$(x-1)(x^2-1)(x^3-1)Q(x) \geq 0$$
for all $x \in \mathbb{R}$

12. Let $Q(x)$ be the monic polynomial with least degree possible which makes
$$(x^3 - 5x^2 + x - 5)(x^2 - 7x + 6)Q(x) \geq 0$$
for all $x > 4$. What is $Q(10)$?

13. **(IMOMath Polynomials P1)** A monic polynomial $f(x)$ of fourth degree satisfies $f(1) = 10$, $f(2) = 20$ and $f(3) = 30$. Determine $f(12) + f(-8)$.

14. **Polynomial division**
For the following choices of $P(x)$ and $D(x)$ do polynomial division and represent
$$P(x) = D(x)Q(x) + R(x)$$
where Q is quotient polynomial and R is remainder with $deg(R) < deg(D)$.

    (a) $P(x) = x^4 + 10x + 1$, $D(x) = x^2 + x + 1$

    (b) $P(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, $D(x) = x^2 + 1$

    (c) $P(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, $D(x) = x^2 - 2$

15. The polynomial P satisfies $P(-2) = 13$, $P(0) = 3$ and $P(2) = 9$. Find the remainder of the polynomial when divided by $x^3 - 4x$.

16. **(AMC12 2001)** The parabola with equation $p(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $q(x) = dx^2 + ex + f$. Which of the following equals $a + b + c + d + e + f$?

    (A) $2b$ (B) $2c$ (C) $2a + 2b$ (D) $2h$ (E) $2k$

17. **(AIME 1983)** What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$?

18. Let $P(x)$ be third degree polynomial with $P(1) = 2, P(2) = 4$ and $P(3) = 6$. If $P(x)$ gives remainder 56 when it is divided by $x + 2$, find the reminder when $P(x + 1)$ divided by $x - 4$

19. Solve the following system in reals:
$$x/y + y/z + z/x = 3$$
$$y/z + z/y + x/z = 3$$
$$x + y + z = 3$$

20. Solve the following system in reals:

$$x - y + z = 6$$

$$x^2 + y^2 + z^2 = 14$$

$$x^3 - y^3 + z^3 = 36$$