ORMC Olympiad Group
Winter: Week 6
Analysis: Functions and Polynomials

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February 5, 2023

Problems

1. Let $f(x) = x^2 - ax + 2020$ where a is a real number. Find a if $f(2020) = f(1048)$.
2. Let $f(x) = x^3 - 4039x^2 + Nx + 1$ where N is an integer. Find the remainder of N when divided by 1000 if $f(2020) = f(2019)$.
3. Let $f(x) = x^2 - 1$ and $g(x) = x - 1$. Find the sum of integers n which does not satisfy
$$(f(g(n))) > g(n-1)$$
4. (HMMT 2005 General 2) Find three real numbers $a < b < c$ satisfying:
$$\begin{aligned} a + b + c &= \frac{21}{4} \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{21}{4} \\ abc &= 1 \end{aligned}$$
5. The polynomial P satisfies $P(1) = 3$, $P(3) = 7$. Find the remainder of the polynomial when divided by $x^2 - 4x + 3$

6. (TNMO-FR 1998) Find the number of primes p, such that $x^3 - 5x^2 - 22x + 56 \equiv 0 \ (mod\, p)$ has no three distinct integer roots in $[0, p)$

7. Polynomial $P(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0$ satisfies $P(n) = 2^n$ for $n = -1000, -999, \ldots, 999, 1000$. Compute the sum of positive even indexed terms, ie compute $a_{2020} + a_{2018} + \cdots + a_2$

8. Find the monic polynomial with least degree which makes

$$(x-1)(x^2-1)(x^3-1)Q(x) \geq 0$$

for all $x \in \mathbb{R}$

9. Let $Q(x)$ be the monic polynomial with least degree possible which makes

$$(x^3 - 5x^2 + x - 5)(x^2 - 7x + 6)Q(x) \geq 0$$

for all $x > 4$. What is $Q(10)$?

10. **(IMOMath Polynomials P1)** A monic polynomial $f(x)$ of fourth degree satisfies $f(1) = 10$, $f(2) = 20$ and $f(3) = 30$. Determine $f(12) + f(-8)$.

11. **Polynomial division**
For the following choices of $P(x)$ and $D(x)$ do polynomial division and represent

$$P(x) = D(x)Q(x) + R(x)$$

where Q is quotient polynomial and R is remainder with $deg(R) < deg(D)$.

(a) $P(x) = x^4 + 10x + 1$, $D(x) = x^2 + x + 1$

(b) $P(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, $D(x) = x^2 + 1$

(c) $P(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, $D(x) = x^2 - 2$

12. The polynomial P satisfies $P(-2) = 13$, $P(0) = 3$ and $P(2) = 9$. Find the remainder of the polynomial when divided by $x^3 - 4x$.

13. **(AMC12 2001)** The parabola with equation $p(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the

parabola with equation $q(x) = dx^2 + ex + f$. Which of the following equals $a + b + c + d + e + f$?

(A) $2b$ (B) $2c$ (C) $2a + 2b$ (D) $2h$ (E) $2k$

14. **(AIME 1983)** What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$?

15. Let $P(x)$ be third degree polynomial with $P(1) = 2, P(2) = 4$ and $P(3) = 6$. If $P(x)$ gives remainder 56 when it is divided by $x + 2$, find the reminder when $P(x + 1)$ divided by $x - 4$

16. Solve the following system in reals:

$$x/y + y/z + z/x = 3$$

$$y/z + z/y + x/z = 3$$

$$x + y + z = 3$$

17. Solve the following system in reals:

$$x - y + z = 6$$

$$x^2 + y^2 + z^2 = 14$$

$$x^3 - y^3 + z^3 = 36$$

18. **(Refail Alizade**) Let $P(x)$ be a third degree polynomial whose all roots are real. P also satisfies the condition that whenever $P(t) = 0$ then $P(t + 1) = 1$. If 7 is a root of $P(x)$ find all possible polynomials.

19. **(NAMO 2010)** $p \neq 0$ and a, b, c are complex roots of the polynomial $x^3 + px + 1 = 0$. Define

$$A = \frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1}$$

What is $p \cdot A$?

20. $a, b, c \in \mathbb{R}$ and the polynomial $p(x) = x^4 + ax^2 + bx + c$ have one root $P(3 + 4i) = 0$. If $P(x) \geq 0$ for all x, find least possible value of c.