Modeling *SET*

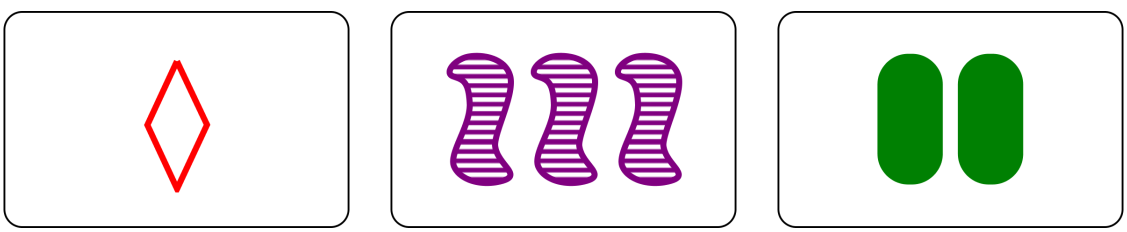
We have finally learned all of the pure math needed to model *SET*! In this packet, we will work through how to do this.

SET as a vector space

First, let's remember how the card game *SET* works. This game has 81 cards. Each card has a design of shapes on the front with four features and three options for each feature. These are

- The color of the shapes shown: red, green, or purple;
- The shape shown: diamond, oval, or squiggle;
- The fill of the shapes shown: empty, shaded, or filled;
- The number of shapes shown: three, one, or two.

Example 1. *Here is an example of the red-diamond-empty-one card, the purple-squiggle-shaded-three card, and the green-oval-filled-two card.*

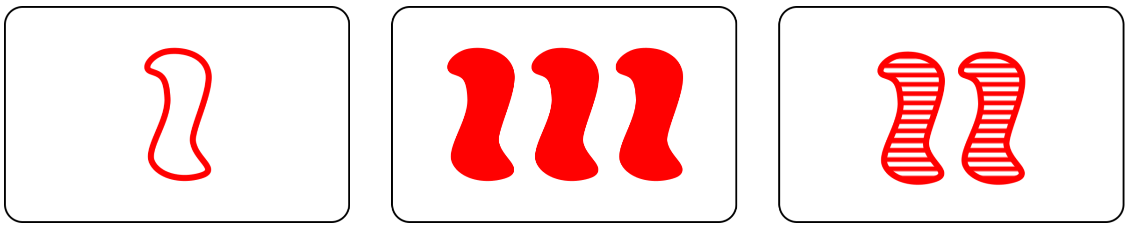


When playing the game, the players place down cards and try to find as many "sets" as a possible. A set is a collection of three cards where:

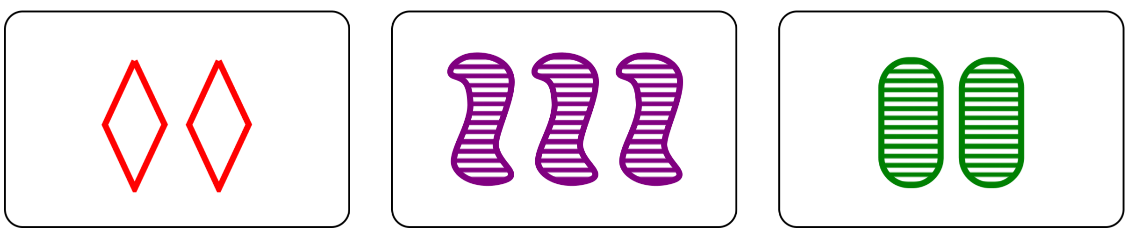
- All three cards are the same number or are all different numbers;
- All three cards are the same color or are all different colors;
- All three cards have the same shading or all have different shading;
- All three cards are the same shape or are all different shapes.

In other words, a set is a collection of three cards so that, for each feature, the attribute is all the same or all different. Notice that the cards in Example 1 form a set. In this case, all the cards show different attributes for each feature.

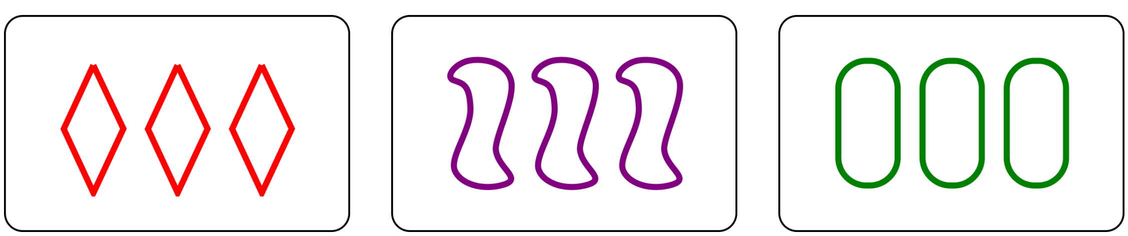
Problem 1. *Do the following cards form a set? If yes, then which features are the same for each card and which features are different?*



Problem 2. *Do the following cards form a set? If yes, then which features are the same for each card and which features are different?*



Problem 3. *Do the following cards form a set? If yes, then which features are the same for each card and which features are different?*



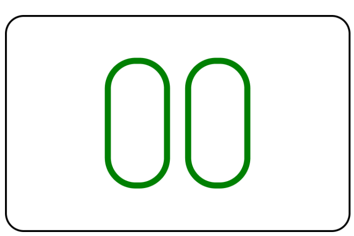
Now, our goal is to be able to write a computer program that can find all of the sets for us in a collection of playing cards. To do this, we need a way of representing each of the 81 cards that the computer can easily understand. Our gut instinct is to try to use numbers somehow, since computers can work very easily with numbers. How about we try to number each of the options for each feature? Then, we can represent each card in the computer as four numbers. In particular, we can write a card as a vector:

$$\begin{bmatrix} \text{code for shape color} \\ \text{code for shape type} \\ \text{code for shape shading} \\ \text{code for shape number} \end{bmatrix}$$

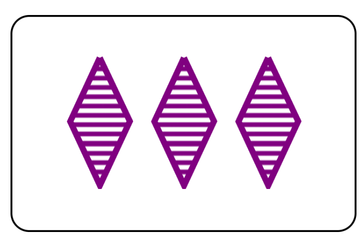
We can use the chart below to determine which number to put in each component of the vector.

	0	**1**	**2**
color	red	green	purple
type	diamond	oval	squiggle
shading	empty	shaded	filled
number	three	one	two

Example 2. *The green-oval-empty-two card and purple-diamond-shaded-three card can be represented by vectors as follows.*



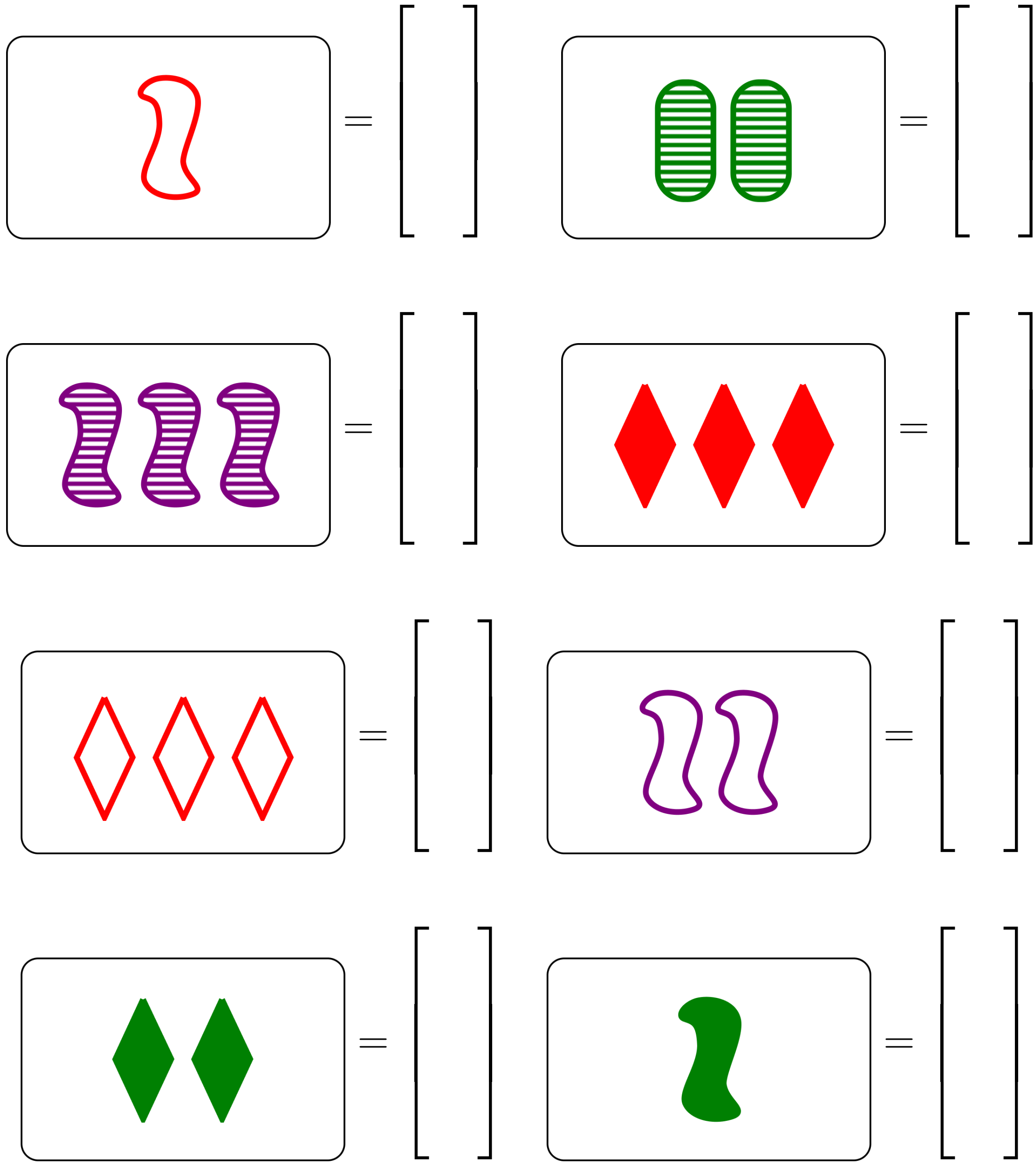
$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$



$$= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Importantly, each card has a unique way of being written as a vector. Moreover, each vector whose entries are 0, 1, or 2 represents a unique card from *SET*. Try out the problems on the following two pages to get used to this vector representation. We add the chart on each page for your reference.

Problem 4. *Fill in the vector representation for each of these cards.*



	0	**1**	**2**
color	red	green	purple
type	diamond	oval	squiggle
shading	empty	shaded	filled
number	three	one	two

Problem 5. *Draw the card corresponding to the vector representation. You can write the color of the card if you do not have colored pens.*

$$\square = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad \square = \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

$$\square = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \square = \begin{bmatrix}2\\1\\0\\0\end{bmatrix}$$

$$\square = \begin{bmatrix}2\\2\\2\\2\end{bmatrix} \qquad \square = \begin{bmatrix}1\\0\\0\\2\end{bmatrix}$$

$$\square = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \qquad \square = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}$$

	0	**1**	**2**
color	red	green	purple
type	diamond	oval	squiggle
shading	empty	shaded	filled
number	three	one	two

By now we are familiar with a specific vector representation for *SET* cards. But what is the point of modeling cards this way? Well, vectors will be easier to store in a computer. More importantly, however, what we want is for this representation to simplify finding sets. The key insight here is that our vectors live inside a vector space! Maybe we can use the structure of this space to find a pattern that sets satisfy. But which vector space do we use?

Problem 6. *Complete the following parts.*

(i) *What is the dimension of a vector representing a SET card? Why?*

(ii) *So, what dimension should we choose for the vector space containing the vectors that represent SET cards?*

(iii) *What are the only numbers that can be in the entries for a vector representing a SET card?*

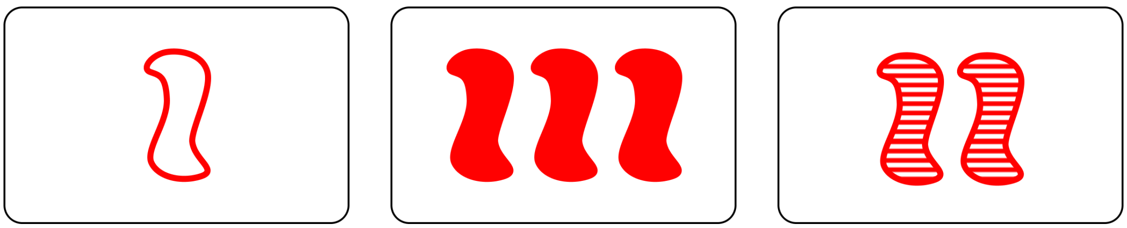
(iv) *List at least five fields that you can think of that contain the numbers in your answer to part (iii).*

(v) *So, combining our answers to part (ii), (iv), list at least five vector spaces that contain all of the vectors which represent SET cards.*

(vi) *Do you have a hunch about which vector space will be the most helpful for us to find sets? List any ideas that you have about why a certain vector space may be best to use. (This question is simply asking for your thoughts and opinions, there are no wrong or right answers.)*

So, $\mathbb{R}^4$, $(\mathbb{Z}_3)^4$, and $(\mathbb{Z}_5)^4$ may be some of the vector spaces that you thought of. These indeed contain all of the vectors that represent *SET* cards since they are four-dimensional vector spaces and the fields $\mathbb{R}$, $\mathbb{Z}_3$, and $\mathbb{Z}_5$ contain the numbers 0, 1, and 2. But which space will be most useful and how can they be useful? Well, now that we have vector spaces we also have vector addition! The easiest thing to check is if the vectors representing cards forming a set add up to something special.

Problem 7. *Consider these cards and complete the following parts.*



(i) *Do these cards form a set?*

(ii) *Write down the vectors representing these cards.*

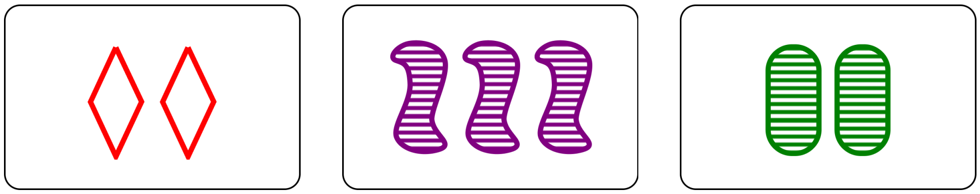
(iii) *What do you get when you add the vectors from part (ii) in* $\mathbb{R}^4$*?*

(iv) *What do you get when you add the vectors from part (ii) in* $(\mathbb{Z}_3)^4$*?*

(v) *What do you get when you add the vectors from part (ii) in* $(\mathbb{Z}_5)^4$*?*

Problem 8. *Complete the following parts.*

(i) *Do the cards below form a set?*

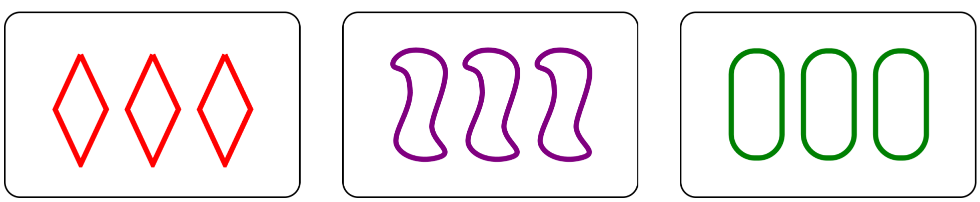


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in $\mathbb{R}^4$? What about in $(\mathbb{Z}_3)^4$? And $(\mathbb{Z}_5)^4$?*

Problem 9. *Complete the following parts.*

(i) *Do the cards below form a set?*

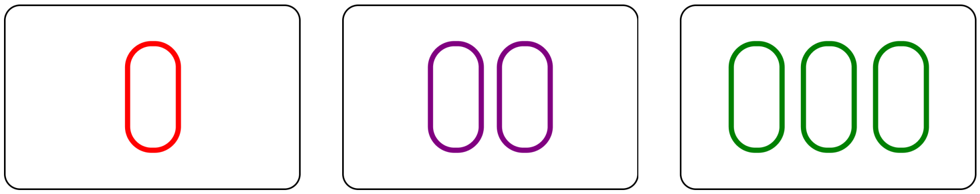


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in $\mathbb{R}^4$? What about in $(\mathbb{Z}_3)^4$? And $(\mathbb{Z}_5)^4$?*

Problem 10. *Complete the following parts.*

(i) *Do the cards below form a set?*

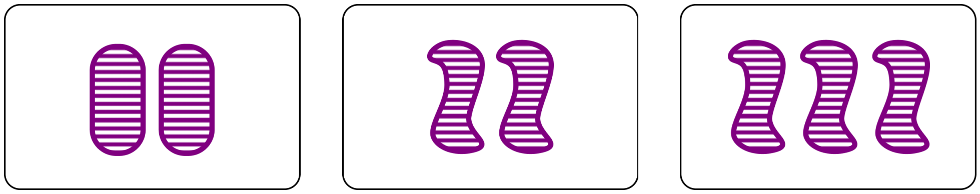


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in* $\mathbb{R}^4$*? What about in* $(\mathbb{Z}_3)^4$*? And* $(\mathbb{Z}_5)^4$*?*

Problem 11. *Complete the following parts.*

(i) *Do the cards below form a set?*

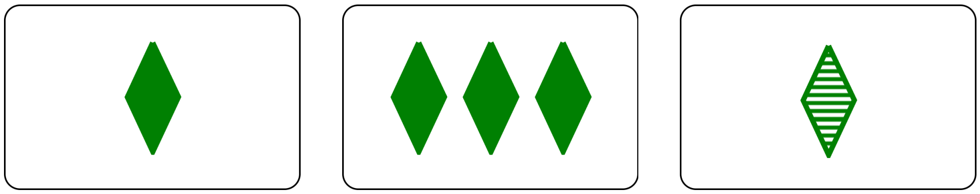


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in* $\mathbb{R}^4$*? What about in* $(\mathbb{Z}_3)^4$*? And* $(\mathbb{Z}_5)^4$*?*

Problem 12. *Complete the following parts.*

(i) *Do the cards below form a set?*

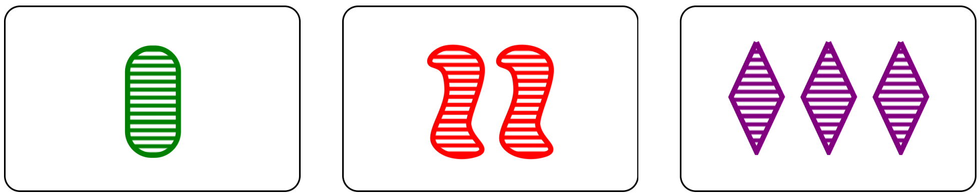


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in* $\mathbb{R}^4$*? What about in* $(\mathbb{Z}_3)^4$*? And* $(\mathbb{Z}_5)^4$*?*

Problem 13. *Complete the following parts.*

(i) *Do the cards below form a set?*

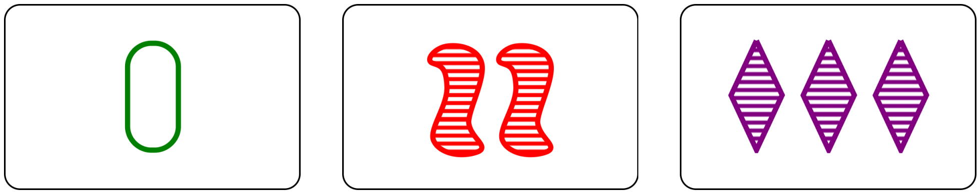


(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in* $\mathbb{R}^4$*? What about in* $(\mathbb{Z}_3)^4$*? And* $(\mathbb{Z}_5)^4$*?*

Problem 14. *Complete the following parts.*

(i) *Do the cards below form a set?*



(ii) *Write down the vectors representing these cards.*

(iii) *What do you get when you add the vectors from part (ii) in $\mathbb{R}^4$? What about in $(\mathbb{Z}_3)^4$? And $(\mathbb{Z}_5)^4$?*

Problem 15. *Overall, do you notice any patterns that the sets have? Do other collections of cards that aren't sets also have this property?*

Problem 16. *So, which vector space do you now think is the most useful for modeling SET? What is special about this vector space as compared to the other two options considered? Did you originally guess this vector space when answering Problem 6?*