ORMC AMC 10/12 Group
Complex Numbers

February 5, 2023

1 Basic Properties

With just the real numbers, equations such as $x^2 = -1$ or $x^2 = y$ for some negative value of y do not have solutions. In order to obtain a solution, we introduce the symbol i, which is a number satisfying $i^2 = -1$.
A complex number is a number of the form $z = a + bi$, where a, b are real numbers. The real part is defined as $a = \mathrm{Re}(z)$ and the imaginary part as $b = \mathrm{Im}(z)$.

We add two complex numbers by summing the real and imaginary parts separately:

- Addition:
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

We multiply together two complex numbers by using the distributive property:

- Multiplication:
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

1.1 Complex Conjugate and Inverses

The **complex conjugate** of z is denoted by $\overline{z}$. If $z = a + bi$, then $\overline{z} = a - bi$. The **magnitude** of a complex number z is denoted by $|z|$ and given by $|z|^2 = z\overline{z} = a^2 + b^2$. This is also the distance from z to the origin. Using these two properties, we can define the inverse and division as follows:

- Inverse:
$$\begin{aligned}(a + bi)^{-1} &= \frac{1}{a + bi} \\ &= \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2}\end{aligned}$$

- Division:
$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}\end{aligned}$$

1.2 Polar Coordinates

Recall that any point in the 2D-plane, $P = (a, b)$, can be represented in polar coordinates as $P = r(\cos\varphi, \sin\varphi)$. Similarly, any complex number $z = a + bi$ can be represented in polar coordinates as $z = r(\cos\varphi + i\sin\varphi)$ where

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\varphi = \arctan\frac{b}{a}.$$

This gives us a way of describing any complex number in terms of its magnitude and angle.

1.3 Examples

1. Let z be a complex number that satisfies $z + 6i = iz$. Find z.

2. Show that for all complex numbers z, $z\overline{z}$ and $z + \overline{z}$ are real numbers.

3. Find the complex number z such that $\frac{z}{1+z} = -1 + z$.

1.4 Exercises

1. A function f is defined by $f(x) = i\overline{z}$. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

2. For what value of n is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?

3. Find c if a, b, c are positive integers which satisfy $c = (a + bi)^3 - 107i$.

2 De Moivre's Theorem

De Moivre's Theorem is a formula for calculating powers of complex numbers. We start with some basic observations. Given some complex number $z = r(\cos\varphi + i\sin\varphi)$, when we try squaring it we get

$$\begin{aligned} z^2 &= (r(\cos\varphi + i\sin\varphi))^2 \\ &= r^2(\cos\varphi + i\sin\varphi)^2 \\ &= r^2((\cos^2\varphi - \sin^2\varphi) + 2i(\cos\varphi\sin\varphi) \\ &= r^2(\cos(2\varphi) + \sin(2\varphi)). \end{aligned}$$

This motivates the conjecture that raising a complex number to the n-th power is equivalent to raising its magnitude to the n-th power and multiplying its angle by n. In other words, we claim that

$$(r(\cos\varphi + i\sin\varphi))^n = r^n\cos(n\varphi) + i\sin(n\varphi).$$

Proof: We proceed with induction. Fix φ and define

$$P(n) := (\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi).$$

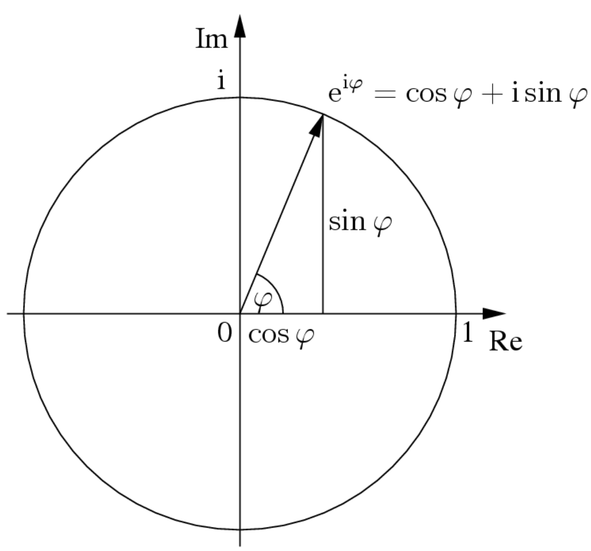
$P(1)$ is true since $\cos\varphi + i\sin\varphi = \cos(1\cdot\varphi) + i\sin(1\cdot\varphi)$. Assume $P(n)$. Then

$$\begin{aligned} (\cos\varphi + i\sin\varphi)^{n+1} &= (\cos\varphi + i\sin\varphi)^n(\cos\varphi + i\sin\varphi) \\ &= (\cos(n\varphi) + i\sin(n\varphi))(\cos\varphi + i\sin\varphi) \\ &= (\cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi) + i(\cos(n\varphi)\sin\varphi + \sin(n\varphi)\cos(\varphi)) \\ &= \cos(n\varphi + \varphi) + i\sin(n\varphi + \varphi) \\ &= \cos((n+1)\varphi) + i\sin((n+1)\varphi). \end{aligned}$$

Thus $P(n) \implies P(n+1)$, so $P(n)$ holds for all $n \geq 1$.

2.1 Euler's Formula

Euler's formula provides a fundamental relationship between trigonometric functions and the complex exponential function e^x. Given some complex number z, suppose that its magnitude is r and its argument is φ. Then Euler's formula states that $z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$.



Euler's formula gives us an alternative proof to De Moivre's Theorem since

$$(e^{i\varphi})^n = e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi).$$

2.2 Examples

1. Using De Moivre's Theorem, find formulas for $\cos 2x$ and $\cos 3x$ in terms of $\cos x$.

2. **(2019 AMC 12B #17)** How many nonzero complex numbers z have the property that $0, z,$ and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

2.3 Exercises

1. **(2000 AIME II # 9)** Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

2. **(2019 AMC 12A #21)** Let
$$z = \frac{1+i}{\sqrt{2}}.$$
What is
$$\left(z^{1^2} + z^{2^2} + z^{3^2} + \cdots + z^{12^2}\right) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \cdots + \frac{1}{z^{12^2}}\right)?$$

3. **(2022 AMC 12A #22)** Let c be a real number, and let z_1 and z_2 be the two complex numbers satisfying the equation $z^2 - cz + 10 = 0$. Points z_1, z_2, $\frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of (convex) quadrilateral $\mathcal{Q}$ in the complex plane. When the area of $\mathcal{Q}$ obtains its maximum possible value, what is the value of c?

4. **(2002 AIME I #12)** Let $F(z) = \dfrac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \dfrac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

3 Roots of Unity

Roots of unity are special complex numbers that, when raised to some integer power, equal 1. For all $n \in \mathbb{N}$, the n-th roots of unity are the n roots of the equation

$$x^n = 1.$$

Since $|x^n| = 1$, all roots of unity have a magnitude of 1. This means that they lie on a unit circle centered at the origin of the complex plane. In fact, we will later show that for any n, the n-th roots of unity lie spaced equally on the unit circle.

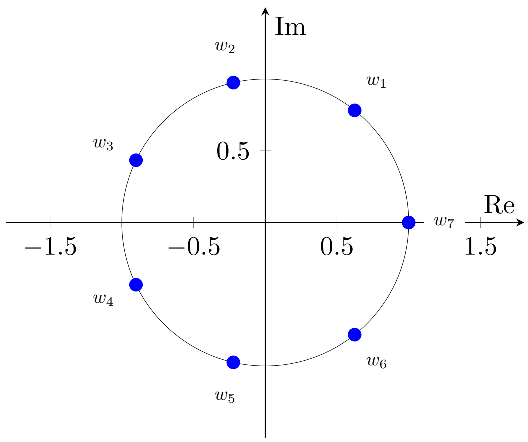


Figure 1: n^{th} roots of unity on $\mathbb{C}$-plane

Proof: Fix $n \in \mathbb{N}$, and let $w = e^{i2\pi/n}$. We will show that the roots of the equation $x^n = 1$ are $1, w, w^2, \ldots, w^{n-1}$. For all $k \in \{0, 1, \ldots, n-1\}$, we have that

$$\begin{aligned}(w^k)^n &= ((e^{i2\pi/n})^k)^n \\ &= ((e^{i2\pi/n})^n)^k \\ &= (e^{i2\pi})^k \\ &= 1.\end{aligned}$$

Thus, for all k, w^k is a solution to $x^n = 1$, and collectively they are the n-th roots of unity. Since consecutive terms differ by an angle of $2\pi/n$, they are spaced equally around the unit circle.

3.1 Examples

1. An equilateral triangle has its centroid located at the origin and a vertex at $(-1, 0)$. What are the coordinates of the other two vertices?

2. There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

3.2 Exercises

1. **(2017 AMC 12B #12)** What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

2. **(2012 AIME I #6)** The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin \frac{m\pi}{n}$, for relatively prime positive integers m and n with $m < n$. Find n.

3. **(2018 AIME I #6)** Let N be the number of complex numbers z with the property that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

4. **(2021 AMC 12A #22)** Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}$, and $\cos \frac{6\pi}{7}$, where angles are in radians. What is abc?