ORMC Intermediate 2A **January 29, 2023**

Oleg Gleizer
Edited by Max Steinberg

Problem 1 *Warm up: think of some other situations where permutations might be helpful (for example, shuffling a deck of cards). Talk with the students next to you about how you might use permutations in these situations and how you might write down the permutations involved.*

Further improving notations

Let us take another look at the permutation μ.

$$\mu = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$$

The permutation does not shuffle the second element. Hence, writing it is redundant. Knowing that the original set consists of four elements, we can write the permutation down as

$$\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$$

Since the second element does not appear in the formula, we know that the permutation does not move it. This convention becomes very convenient with larger permutations. For example, let us take another look at Sam Loyd's formulation of the 15 puzzle. Since we need to keep track of the empty square, as well as of the numbered ones, let us consider it as the 16th tile.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix}$$

switches the 14th and 15th elements only. Writing down the 14 elements it does not move is a waste of time! In the new notations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 14 \end{pmatrix}.$$

Since all other elements are not mentioned, we know that the permutation does not shuffle them.

Here is one more example. Let $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ be a permutation of six elements. Since the elements 2, 5, and 6 are not listed, μ keeps them in place. So in fact, $\mu = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}$.

Problem 2 *The permutation* $\nu = \begin{pmatrix} 3 & 1 \end{pmatrix}$ *acts on a set of three elements. Write down its full version.*

$$\nu =$$

What is the order of ν*?*

Write down the short form of $\nu^{-10{,}000{,}831}$.

$$\nu^{-10{,}000{,}831} =$$

Problem 3 *The permutation* $\delta = \begin{pmatrix} 3 & 5 & 7 & 1 \end{pmatrix}$ *acts on a set of seven elements. Write down its full version.*

$$\delta =$$

What is the order of δ*?*

Write down the short form of $\delta^{-10{,}000{,}000}$.

$$\delta^{-10{,}000{,}000} =$$

A permutation that swaps two elements and doesn't shuffle anything else is called a *transposition*. For example, the permutation that switches the order of the third and fifth element in a six-element set is $\begin{pmatrix} 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 4 & 3 & 6 \end{pmatrix}$.

Problem 4 *What is the inverse of the transposition* $\begin{pmatrix} 5 & 3 \end{pmatrix}$?

$$\begin{pmatrix} 5 \ 3 \end{pmatrix}^{-1} =$$

Problem 5 *What is the order of any transposition?*

Any permutation can be realized as a product of transpositions. For example, let us consider the permutation
$\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ from before. Applying the transposition $\begin{pmatrix} 2 & 1 \end{pmatrix}$ to the original order of the elements gives us the following.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 & 4 \end{pmatrix}$$

Let us apply the transposition $\begin{pmatrix} 4 & 1 \end{pmatrix}$ to the result.

$$\begin{pmatrix} 2 & 1 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix}$$

Finally, applying the transposition $\begin{pmatrix} 3 & 1 \end{pmatrix}$ finishes the job.

$$\begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$$

Or more concisely,

$$\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}.$$

Problem 6 *Realize the permutation* $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$ *as a product of transpositions.*

Problem 7 *Write down the permutation μ that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

$\longrightarrow$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	
13	**15**	**14**	**12**

$$\mu =$$

The problem continues on the next page.

Find the product $\mu \circ \begin{pmatrix} 15 & 14 \end{pmatrix}$ and compare the answer to the order of the squares on the second picture of the previous page.

$$\mu \circ \begin{pmatrix} 15 & 14 \end{pmatrix} =$$

Problem 8 *Write down the permutation that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	
13	**15**	**14**	**12**

$\longrightarrow$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

Problem 9 *Find the order of the permutation* $\sigma = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \end{pmatrix}$.

Problem 10 *Without doing any more computations, find the following for the permutation* $\sigma = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ *from Problem 9.*

$$\sigma^{-1} =$$

$$\sigma^{126} =$$

Parity of a permutation

If a permutation σ moves the element in the position i to the position k, we write $\sigma(i) = k$. Let us consider the permutation σ from Problems 9 and 10 one more time. It moves the fifth element to the first position, so $\sigma(5) = 1$. It moves the first element to the second position, so $\sigma(1) = 2$.

Problem 11 *For the permutation* σ *from Problems 9 and 10, find the following.*

$$\sigma(2) =$$

$$\sigma(3) =$$

$$\sigma(4) =$$

If $i < j$, but $\sigma(i) > \sigma(j)$, then the pair (i, j) is called an *inversion* of the permutation σ. In other words, a inversion of a permutation is a smaller number moved to the right of a larger number (or a larger number moved to the left of a smaller number). For example, the permutation $\sigma = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ from Problems 9, 10, and 11 moves 5 to the first position, so $(5, 1)$, $(5, 4)$, $(5, 3)$, and $(5, 2)$ are all inversions of σ.

Note 1 *Although the words "inverse" and "inversion" are very similar, the notions of an inverse of a permutation and an inversion of a permutation are very different! An inverse of a permutation σ is the permutation σ^{-1} that undoes what the original permutation σ does. The inversion of a permutation σ is a disorder the permutation σ creates.*

Problem 12 *Write down all other inversions of the permutation* $\sigma = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \end{pmatrix}$.

The *sign* of a permutation is defined according to the following formula.

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)} \tag{1}$$

where $N(\sigma)$ is the number of inversions of the permutation σ. For example, the total number of inversions of the permutation σ from Problems 9, 10, 11, and 12 is seven (check it!), so $\operatorname{sgn}(\sigma) = (-1)^7 = -1$.

Problem 13 *What is the sign of the trivial permutation?*

$$sgn(e) =$$

Problem 14 *Find the signs of the following permutations.*

$$sgn\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} =$$

$$sgn\begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix} =$$

Problem 15 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

Recall that a transposition (ji) is a permutation that changes the positions of only two elements, i-th and j-th.

Theorem 1 *The sign of any transposition is* -1.

Before giving Theorem 1 a formal proof, let us check a few cases.

Problem 16 *What is the sign of the transposition* $\sigma = (52)$ *acting on a set of five elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (52)$ *acting on a set of six elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (63)$ *acting on a set of seven elements?*

$$sgn(\sigma) =$$

To prove Theorem 1, let us first observe that a transposition of two neighbouring elements, called an *adjacent transposition*, always changes the number of inversions by one. Let us consider the transposition $\delta = (i+1, i)$.

All the elements except for the $i+1$-st that formed inversions with the i-th element still form inversions with it when it moves to the $i+1$-st position. All the elements except for the i-th that formed inversions with the $i+1$-st one keep doing so when the latter moves one position to the left. If the pair $(i, i+1)$ formed an inversion, δ removes it. If the pair formed no inversion, δ creates one.

The following Lemma finishes the proof of Theorem 1.

Lemma 1 *Any transposition can be realized as a product of an odd number of adjacent transpositions.*

Proof — Consider the transposition (ji) where $j > i+1$. The following product of $j-i-1$ adjacent transpositions

$$(j-1, j-2) \circ \ldots \circ (i+2, i+1) \circ (i+1, i)$$

moves the i-th element to the $j-1$-st position one step at a time. The adjacent transposition

$$(j, j-1)$$

swaps it with the j-th element. Finally, the following product of $j-i-1$ adjacent transpositions

$$(i+1, i) \circ (i+2, i+1) \circ \ldots \circ (j-1, j-2)$$

moves the element that was originally in the j-th position to the i-th. This way, any transposition (ji) where $j > i+1$ can be represented as a product of $2(j-i-1)+1$ adjacent transpositions. $\square$

Example 1

$$(52) = (32) \circ (43) \circ (54) \circ (43) \circ (32)$$

Problem 17 *Represent the transposition* (63) *as a product of adjacent transpositions.*

$$(63) =$$

Is the number of the adjacent transpositions odd or even?

The permutations that have the sign 1 are called *even*. The permutations that have the sign -1 are called *odd*. This way, all permutations are split into two classes. A class of a permutation is called its *parity*. Theorem 1 proves that transpositions are odd permutations and that multiplying a permutation by a transposition changes the parity of the former.

Problem 18

- *Find the sign of the permutation* $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ *acting on a set of five elements.*

$$sgn(\mu) =$$

- *Find the product* $(51) \circ \mu$.

$$(51) \circ \mu =$$

- *Find the sign of the permutation* $(51) \circ \mu$.

$$sgn\left((51) \circ \mu\right) =$$

Note that Theorem 1 gives a different way to compute the sign of a permutation. Instead of counting inversions, let us decompose the permutation into a product of transpositions. Then the sign of the transposition is

$$(-1)^{\text{the number of transpositions in the product}}. \tag{2}$$

Various representations of a permutation as a product of transpositions can have different length, but they always have the same parity.

Every move of the 15 puzzle is a transposition of a special type. You swap a square numbered one through fifteen with the empty square (originally in the 16th position). This observation alone is not enough to prove that the 15 puzzle configuration suggested by Sam Loyd has no solution.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Problem 19 *Write down all the inversions of the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$.

What is the sign of the permutation?

$$sgn(\sigma) =$$

Problem 20 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

Problem 21 *Represent the transposition* (41) *as a product of adjacent transpositions.*

$$(41) =$$

Problem 22 *Represent the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$ *from Problem 19 as a product of transpositions.*

$$\sigma =$$

Use the formula

$$sgn(\sigma) = (-1)^{\#t} \tag{3}$$

where $\#t$ *is the number of transpositions in the product to find the sign of* σ*. Compare your answer to that of Problem 19.*

$$sgn(\sigma) =$$

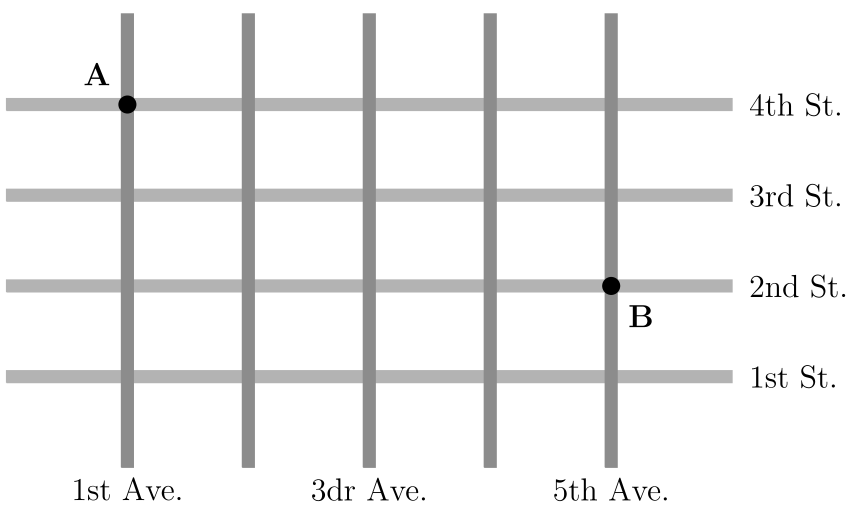
Did you expect the answers to be the same? Why or why not?

There are two tools needed to prove that the 15 puzzle configuration suggested by Sam Loyd has no solution.

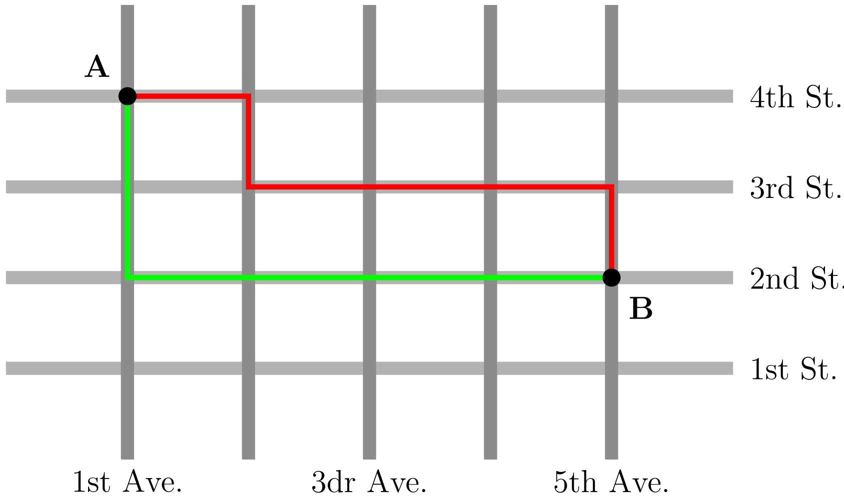
1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Taxicab geometry

Imagine that you take a taxicab to get from point A to point B in a city with streets and avenues forming a rectangular pattern.



Similar to Euclidean geometry, there exists a shortest path. Unlike Euclidean geometry, the shortest path is not unique. For example, the green and red routes on the picture below are both shortest ways from A to B.



Problem 23 *On the picture above, draw a third shortest path from A to B.*

The point A lies at the intersection of the 1st Ave. and the 4th St. Let us write this fact down as follows.

$$A = (1, 4)$$

B lies at the intersection of the 5th Ave. and the 2nd St.

$$B = (5, 2)$$

Let a be the distance between two neighbouring avenues and let s be the distance between two neighbouring streets. No matter what shortest path the cab driver chooses, he needs to drive 4 blocks East and 2 blocks South.

$$d_{tc}(A, B) = 4a + 2s$$

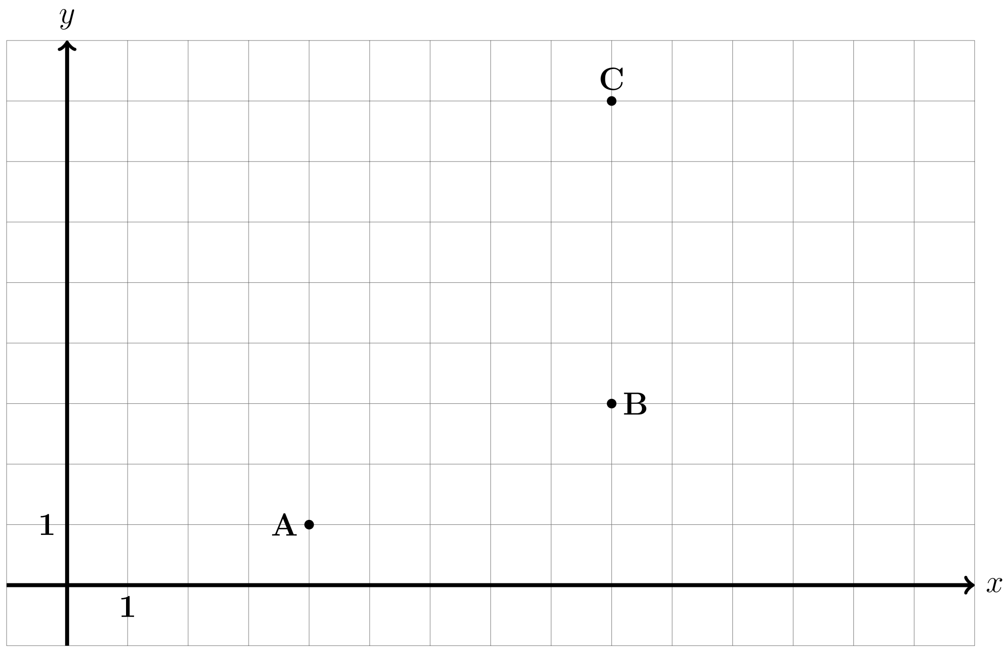
Problem 24 *Find the Euclidean distance $d_E(A, B)$ between the points A and B.*

$$d_E(A, B) =$$

Without doing any computations, put the correct sign, >, <, or =, between the distances below. Explain your choice.

$$d_E(A, B) \qquad\qquad d_{tc}(A, B)$$

Problem 25 *For the grid below, $a = s = 1$.*



Find the following taxicab distances.

$$d_{tc}(A, B) =$$

$$d_{tc}(A, C) =$$

$$d_{tc}(B, C) =$$

If we use the taxicab distance instead of the Euclidean one, would the triangle inequality hold for the triangle ABC?

For any two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the coordinate plane, let us define the taxicab distance between them as follows.

$$d_{tc}(A, B) = |x_1 - x_2| + |y_1 - y_2| \tag{4}$$

Problem 26 *Find the taxicab distance between the points* $A = (-2, 7)$ *and* $B = (3, -5)$.

$$d_{tc}(A, B) =$$

Problem 27 *The taxicab distance between the points* A *and* B *is zero.*

$$d_{tc}(A, B) = 0$$

Can the points be different? Why or why not?

Note that the taxicab distance shares some basic properties with the Euclidean one. The distance from A to B equals the distance from B to A.

$$d_{tc}(A, B) = d_{tc}(B, A) \qquad\qquad d_E(A, B) = d_E(B, A)$$

In both cases, the distance between two points is zero if and only if the points coincide.

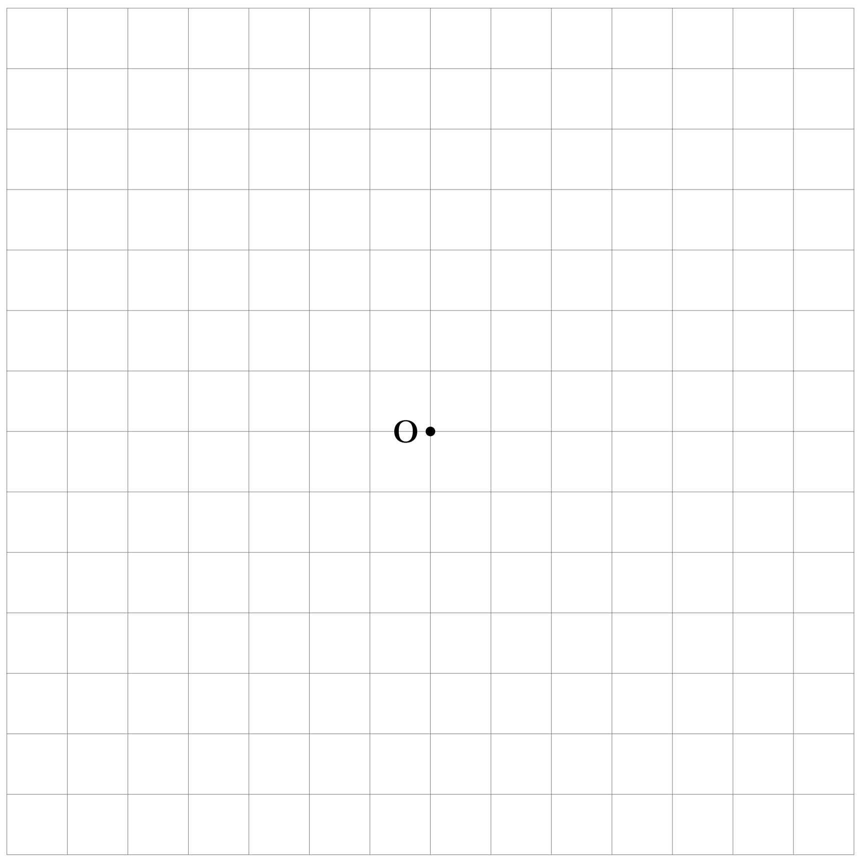
$$d_{tc}(A, B) = 0 \;\Leftrightarrow\; A = B \;\Leftrightarrow\; d_E(A, B) = 0$$

The distance between two different points is always positive.

$$A \neq B \;\Rightarrow\; d_{tc}(A, B) > 0 \;\text{ and }\; d_E(A, B) > 0$$

We have observed one difference between the distances in Problem 25. There, $d_{tc}(A, C) = d_{tc}(A, B) + d_{tc}(B, C)$. For the Euclidean distance, this means that the point B lies on the straight line AC between the points A and C, quite obviously not necessarily the case for the taxicab distance.

Problem 28 *On the grid below, mark all the points that have the taxicab distance 6 from the point O.*



Problem 29 *Give the definition of a circle of radius R centred at the point O in the space below.*

Was the figure constructed in Problem 28 a circle? Why or why not?

Problem 30 *Find the taxicab distance from the current position of the empty square to the lower-right corner of the 15 puzzle.*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

Finally, we have all the tools we need to prove that Sam Loyd's configuration is unsolvable.

Let $\mathcal{P}$ be a function that assigns each configuration $\mathcal{C}$ of the 15 puzzle one of the two values, either zero or one. Let us set $\mathcal{P}(\mathcal{C}) = 0$ if the the sum of the inversions of $\mathcal{C}$ plus the taxicab distance from its empty square position to the lower-right corner of the puzzle is an even number. Let us set $\mathcal{P}(\mathcal{C}) = 1$ otherwise. For example, let us take another look at the configuration $\mathcal{C}$ we have considered in Problem 30.

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

The number of inversions of this configuration is 16. The taxicab distance from the empty square of the configuration to the lower-right corner is 3. The sum, $16 + 3 = 19$, is an odd number, so $\mathcal{P}(\mathcal{C}) = 1$.

Let us call a configuration $\mathcal{C}$ of the 15 puzzle *even*, if $\mathcal{P}(\mathcal{C}) = 0$ and let us call it *odd* otherwise. This way, all the configurations of the puzzle are split into two classes, even and odd.

Problem 31 *Is the following configuration*

1	**2**	**3**	**4**
5		**6**	**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

even or odd? Try to answer this question without doing too many calculations. Hint: compare this configuration to that on page 23.

The following theorem answers the solvability question that has opened this mini-course.

Theorem 2 *Odd configurations of the 15 puzzle are not solvable.*

Indeed, Sam Loyd's configuration $\mathcal{C}$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

has only one inversion, $(15, 14)$. The taxicab distance from the empty square position of the configuration to the lower-right corner is zero. Hence, $\mathcal{P}(\mathcal{C}) = 1 + 0 = 1$. The configuration is odd and thus, according to Theorem 2, has no solution.

Proof of Theorem 2 — Each move of the 15 puzzle is a transposition that swaps a square numbered 1 through 15 with the empty square. According to Theorem 1, a transposition always changes the number of inversions of a permutation by an odd number.

Any move of the 15 puzzle changes the taxicab distance from the current empty square position to the lower-right corner by one. Hence, the sum of the changes of the number of the inversions and of the taxicab distance in consideration is always an even number.

The winning configuration of the 15 puzzle, the one corresponding to the trivial permutation of the sixteen element, is even. According to the above, it cannot be obtained from an odd configuration. □

Theorem 3 *Any even configuration of the 15 puzzle is solvable.*

Theorem 3 is not hard to prove using mathematical induction. We are not going to do it at the moment. The following theorem is much harder to prove.

Theorem 4 *Lengths of the optimal solutions of the 15 puzzle range from 0 to 80 single-tile moves.*

Review session

Problem 32

- *What is the number of inversions of the permutation corresponding to the following configuration of the 15 puzzle?*
(Remember, the empty square is considered as the 16th tile.)

1	**2**	**3**	**4**
	5	**6**	**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

$$\#\ \textit{of inversions} =$$

- *What is the taxicab distance from the current position of the empty square to the lower-right corner of the puzzle?*

$$d_{tc} =$$

- *What is the value $\mathcal{P}(\mathcal{C})$ of the invariant $\mathcal{P}$ for the configuration $\mathcal{C}$ above?*

$$\mathcal{P}(\mathcal{C}) =$$

- *Is the above configuration $\mathcal{C}$ of the 15 puzzle solvable? Why or why not?*

Problem 33 *Without doing any extensive computations, decide whether the following configuration of the 15 puzzle is solvable. Hint: compare this configuration to the one on page 26 (the problem continues to the next page).*

1	**2**	**3**	**4**
	5	**6**	**8**
9	**10**	**7**	**12**
13	**15**	**14**	**11**

Explain your decision.

Problem 34 *Without doing any extensive computations, decide whether the following configuration of the 15 puzzle is solvable. Hint: compare it to the winning configuration.*

2	**1**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Explain your decision.

Problem 35 *Find the products $\delta \circ \sigma$ and $\sigma \circ \delta$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix}$$

$$\delta \circ \sigma = \begin{pmatrix} \qquad\qquad\qquad\qquad \end{pmatrix}$$

$$\sigma \circ \delta = \begin{pmatrix} \qquad\qquad\qquad\qquad \end{pmatrix}$$

Do the permutations σ and δ commute?

Problem 36 *The order of the permutation σ is 4. Find the following.*

$$\sigma^{444}(7) =$$

Problem 37 *For the transposition $\sigma = (52)$, find the following.*

$$\sigma^{-1} =$$

$$\sigma^{2014} =$$

$$sgn\left(\sigma^{2015}\right) =$$

Problem 38 *Find the order of the permutation* $\sigma = \begin{pmatrix} 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$.

$$The\ order\ of\ \sigma =$$

Without doing any additional computations, find the following.

$$\sigma^{-2} = \begin{pmatrix} & & & & & & & & & & \end{pmatrix}$$

Problem 39 *Represent the transposition* (85) *as a product of adjacent transpositions.*

$$(85) =$$

Problem 40

- *Write down the full form of the permutation* $\mu = \begin{pmatrix} 5 & 7 & 3 \end{pmatrix}$ *acting on a set of nine elements.*

$$\mu = \Bigg(\qquad\qquad\qquad\qquad\qquad\qquad \Bigg)$$

- *Find the sign of* μ.

$$sgn(\mu) =$$

- *Find* μ^{-1} *and write it down in the short form.*

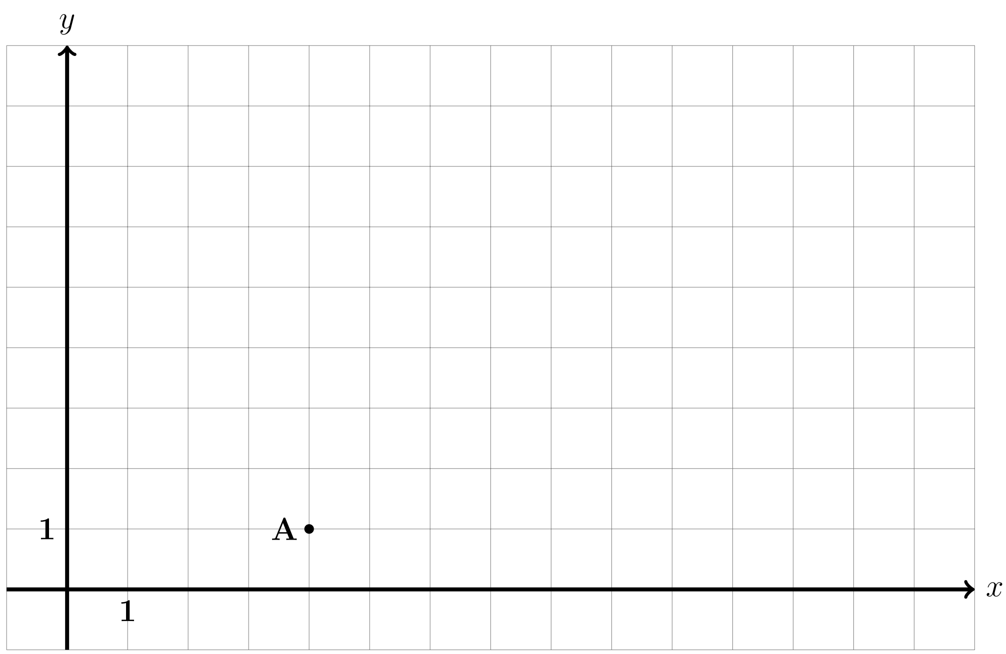
$$\mu^{-1} = \Bigg(\qquad\qquad\qquad \Bigg)$$

Problem 41

- *On the grid below, draw a point B such that* $d_{tc}(A,B) > d_E(A,B)$.

- *On the grid below, draw a point C such that* $d_{tc}(A,C) = d_E(A,C)$.

- *On the grid below, draw a point D such that* $d_{tc}(A,D) < d_E(A,D)$. *Is it possible? Why or why not?*



Problem 42 *Find the taxicab distance between the points* $A = (p,q)$ *and* $B = (x,y)$.

$$d_{tc}(A, B) =$$

Find the taxicab distance between the points $C = (-3.3, 5.2)$ *and* $D = (-2.7, -7.8)$.

$$d_{tc}(C, D) =$$

Problem 43 *For the configuration of the 15 puzzle below, find the taxicab distance between the squares 6 and 11.*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

$$d_{tc}(6, 11) =$$

Problem 44

- *What is the number of inversions of the permutation corresponding to the following configuration of the 15 puzzle?*
(Remember, the empty square is considered as the 16th tile.)

	2	**3**	**4**
1	**5**	**6**	**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

$\#$ *of inversions* $=$

- *What is the taxicab distance from the current position of the empty square to the lower-right corner of the puzzle?*

$$d_{tc} =$$

- *What is the value* $\mathcal{P}(\mathcal{C})$ *of the invariant* $\mathcal{P}$ *for the configuration* $\mathcal{C}$ *above?*

$$\mathcal{P}(\mathcal{C}) =$$

- *Is the above configuration* $\mathcal{C}$ *of the 15 puzzle solvable? Why or why not?*