OLGA RADKO MATH CIRCLE: ADVANCED 3

JOAQUÍN MORAGA

Solutions 7: Polynomial Rings

Solution 7.1:

i) $x^6 + x^3 + 1$, let $\omega = x^3$. Then we get: $\omega^2 + \omega + 1$. This has no roots in the real numbers, but we have complex solutions: $\omega = -1/2 + -\sqrt{3}/2$. So, $x^3 = \cos \pi/3 + \sin \pi/3$, or $x^3 = \cos 2\pi/3 + \sin 2\pi/3$. Notice that x^3 belongs to the unit circle in the complex plane. So, since I can treat $z = \cos\theta + i \sin\theta$ as a rotation by θ, then I want: $3\theta_1 = \pi/3 + 2k\pi$ and $3\theta_2 = 2\pi/3 + 2k\pi$ Solving for θ_1, θ_2, I get:
$x_1 = \cos \pi/9 + i \sin \pi/9$
$x_2 = \cos 7\pi/9 + i \sin 7\pi/9$
$x_3 = \cos 13\pi/9 + i \sin 13\pi/9$
$x_4 = \cos 2\pi/9 + i \sin 2\pi/9$
$x_5 = \cos 8\pi/9 + i \sin 8\pi/9$
$x_6 = \cos 14\pi/9 + i \sin 14\pi/9$

ii) $x^8 + 1 > 0$, so there are no solutions in the real numbers. In the complex numbers, we have once again $x^8 = -1$, so similarly as before: we are looking for an angle θ such that $8\theta = \pi + 2k\pi$. Solving for θ we get:
$x_k = \cos{(2k + 1)\pi/8} + i \sin{(2k + 1)\pi/8}$, with $k \in N$ ranging from 0 to 7.

iii) $x^4 + x^2 + 1 > 0$, so again we have no real roots. Set, $\omega = x^2$, and we get: $\omega^2 + \omega + 1$. As in the first problem we have $\omega_1 = \cos \pi/3 + i \sin \pi/3$ and $\omega_2 = \cos 2\pi/3 + i \sin 2\pi/3$ So, since $x^2 = \omega$ we want angles θ_1, θ_2 such that $2\theta_1 = \pi/3 + 2k\pi$, and $2\theta_2 = 2\pi/3 + 2k\pi$ The corresponding complex numbers then are:
$x_1 = \cos \pi/6 + i \sin \pi/6$
$x_2 = \cos 7\pi/6 + i \sin 7\pi/6$
$x_3 = \cos \pi/3 + i \sin \pi/3$
$x_4 = \cos 4\pi/3 + i \sin 4\pi/3$

Solution 7.2:

i) $x^4 + x^3 + x^2 + x + 1 = q(x)$ in $Z/5Z$ Since there are only 5 elements in $Z/5Z$ we can easily check: $q(0) = 1$, $q(1) = 0$, $q(2) = 1$, $q(3) = 1$, $q(4) = 1$.

ii) $x^{12} - x^2 + 1 = q(x) in Z/13Z$. By Fermat's little theorem: for any integer a, we have $a^{n-1} \equiv 1 mod n$. So, in $Z/13Z$, $x^{12} = 1, \forall x \in Z/13Z$ So, the equation is equivalent to: $2 - x^2 = 0$. Notice that in Z/nZ $a^2 = (n-a)^2$, since $(n-a)^2 = n(n-2a) + a^2 \equiv a^2 mod n$. So, we only need to check half the elements in Z/13Z. Also, we don't need to check below $\sqrt{13}$. (why?).
So, $q(4) = 12$, $q(5) = 3$, $q(6) = 5$. No solutions in Z/13Z.

iii) $q(x) = x^{16} + x^3 + 1 \in Z/17Z$, as before from Fermat's little theorem, we have $x^{16} = 1$, so our equation is equivalent to: $x^3 + 2 = 0$. Notice also: $(17 - x)^3 \equiv -x^3 mod 17 \equiv 17 - x^3 mod 17$. So, once we compute x^3 we can proceed to compute $q(x)$ and $q(17-x) = (17-x)^3 + 2 = 2 - x^3$. Further, similarly as before we don't need to check numbers less than $\sqrt[3]{17}$.
$3^3 = 10$ so $q(3) = 12$ and $q(14) = 9$, if we continue in the same manner we find that only $q(9)$ gives us 0.

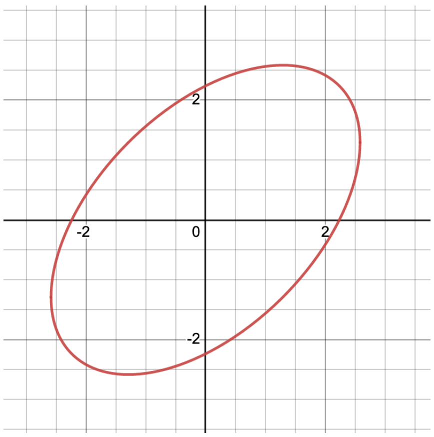
Solution 7.3:

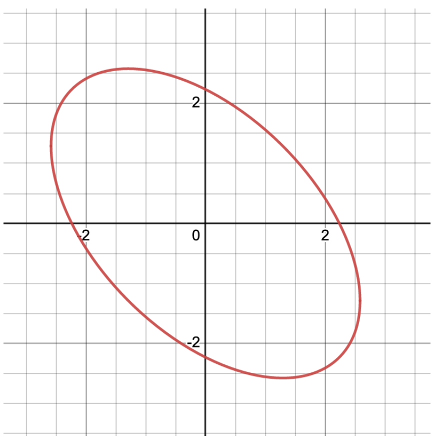
(1) Given $p(x, y) = x^2 + y^2$, $q(x, y) = x^2 - y^2$
  (a) $p + q = x^2 + y^2 + x^2 - y^2 = 2x^2$
  (b) $pq = (x^2 + y^2)(x^2 - y^2) = x^4 - y^4$
(2) Given $p(x, y) = x - y, q(x, y) = x^3 + y^3$
  (a) $p + q = x^3 + x - y + y^3$
  (b) $pq = x^4 + xy^3 - x^3y - y^4$
(3) Given $p(x, y) = xy, q(x, y) = x^2 + xy + y^2$
  (a) $p + q = x^2 + 2xy + y^2$
  (b) $pq = x^3y + x^2y^2 + xy^3$

Solution 7.4:

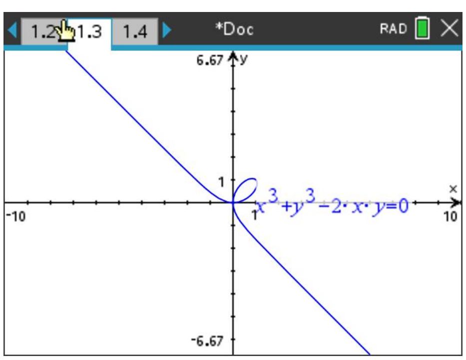
(1) Given $p(x, y, z) = x^2 + y^2 + z^2$, $q(x, y, z) = x^2 - y^2 + z^2$
  (a) $p + q = 2x^2 + 2z^2$
  (b) $pq = (x^2 + z^2)^2 - (y^2)^2 = (x^4 - y^4) + (2x^2)z^2 + z^4$
(2) Given $p(x, y, z) = x - y + z, q(x, y, z) = x^3 + y^3 + z^3$
  (a) $p + q = x + y - z + x^3 + y^3 + z^3$
  (b) $pq = (x^4 + xy^3 - x^3y - y^4) + (x^3 + y^3)z + (x - y)z^3 + z^4$
(3) Given $p(x, y, z) = xyz, q(x, y, z) = x + y + z$
  (a) $p + q = xyz + x + y + z$
  (b) $pq = (x^2y + xy^2)z + (xy)z^2$

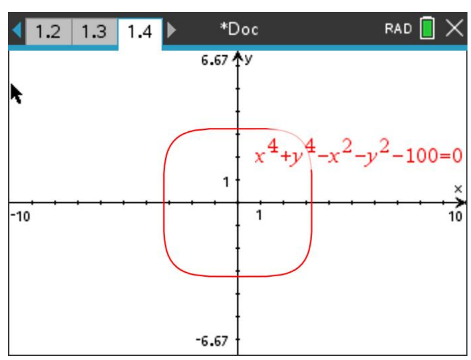
Solution 7.5:





Solution 7.6:
See the following two pictures.





Solution 7.7:

We want to seek ordered pairs (x, y) with $x, y \in \mathbb{Z}_3$ such that $x^{10} + y^{10} + xy = 0$. Note that if $z = 1, 2$, then $z^{10} = 1$, while $0^{10} = 0$. Thus, if one of x, y is 0, the other must be 0 too since $xy = 0$ always. Due to the symmetry in the polynomial, we only need then to check if $(1, 1), (1, 2), (2, 2)$ work. By simply plugging in the values and checking, we see that we have three solutions to the equation in total: $(0, 0), (1, 1), (2, 2)$. The vanishing locus is just the set of these three points.
The second polynomial can be dealt with similarly. Note that $z^5 = z$ for all z by Fermat's little theorem. Again, by symmetry, we only need to check if (x, y) for which $x \leq y$ work. The answer is $\{(0, 0), (0, 2), (0, 3), (2, 0), (3, 0)\}$.

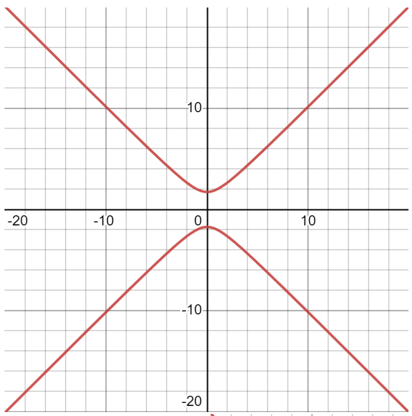
Solution 7.8:
There is no such polynomial. Assume towards contradiction that there is such a polynomial $p(x, y)$. Then consider $f(x) := p(x, -1)$, a polynomial in independent variable x only. Then assume that the polynomial has degree n; by fundamental theorem of algebra, there should be exactly n zeros in $\mathbb{C}$. However, we can select $n+1$ points arbitrarily on the interval $[-1, 1]$ so that $f(x) = p(x, -1) = 0$, based on the fact that $p(x, y)$ is zero on the square S, and particularly on the left side of the square. So we reach a contradiction.

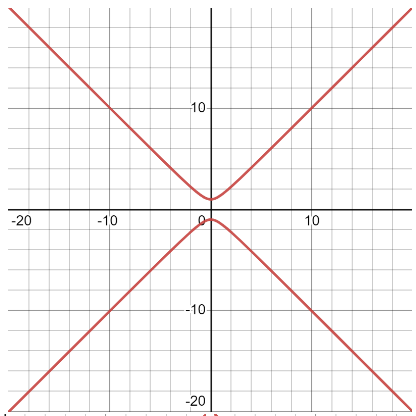
Solution 7.9:
The plots are shown below. The graphs are in the order of $t = -3, -1, 0, 1, 3$, from top left to bottom right. Therefore, we can conclude that as time goes to infinity, the vanishing set will be a pair of hyperbola, open to left and right, and seperate more and more.

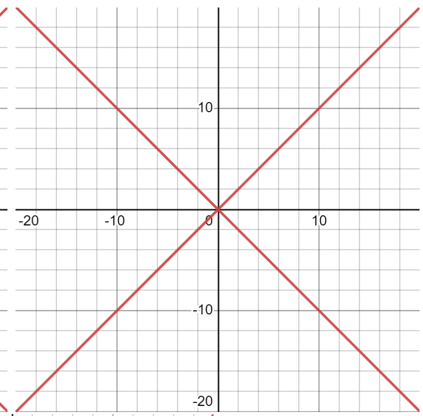
Solution 7.10:
For the polynomial $p(x, y, z) = z - x^2 + y^2$, the horizontal cross sections of the vanishing set are the vanishing sets from 7.9 when t is set equal to z.

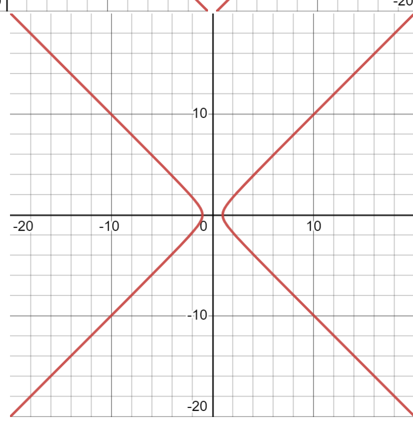
Solution 7.11:
Yes, for any two points in $\mathbb{R}^2$, (x_1, y_1) and (x_2, y_2) the line $(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0$ goes through the points where x_1, x_2, y_1, and y_2 are constants.
For any 4 points in $\mathbb{R}^2$, there is a conic of the form $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ where the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ each contain 2 of the points. This means you could find a conic passing through every 3 points.

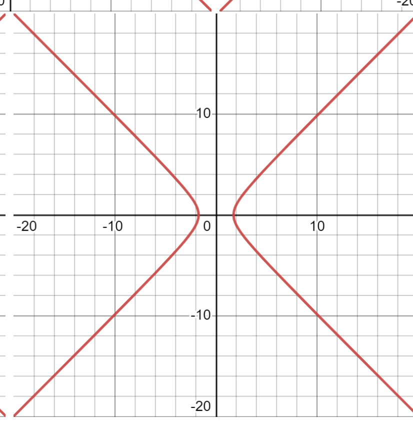
UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: jmoraga@math.ucla.edu


10
-20
-10
0
10
-10
-20


10
-20
-10
0
10
-10
-20


10
-20
-10
0
10
-10
-20


10
-20
-10
0
10
-10
-20


10
-20
-10
0
10
-10
-20