OLGA RADKO MATH CIRCLE: ADVANCED 3

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Worksheet 11: Affine Varieties II

The following problem says that maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ corresponds to points in $\mathbb{C}^n$.

Problem 11.0: Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Prove that I is a maximal ideal if and only if $V(I)$ is a point in $\mathbb{C}^n$.

Show that in $\mathbb{R}[x_1, \ldots, x_n]$ there are ideals I that are not maximal for which $V(I)$ is a single point. What is the least number of equations that you can find for such an I?

Solution 11.0:

Let $p \in \mathbb{C}^n$ be a point with coordinates $(c_1, \ldots, c_n) \in \mathbb{C}^n$. Let $m_p := \langle x_1 - c_1, \ldots, x_n - c_n \rangle$ be the associated maximal ideal.

Problem 11.1: Let $V(I)$ be the affine variety in $\mathbb{C}^n$ corresponding to the radical ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. Show that the affine variety $V(I)$ contains the point $p \in \mathbb{C}^n$ if and only if the ideal I is contained in m_p.

Solution 11.1:

A linear subspace of $\mathbb{C}^n$ is a variety defined only by linear equations. The *dimension* of the linear subspace is n minus the least number of linear equations that we need to generate the ideal. Linear subspaces of dimension 0 are called *points*, of dimension 1 are called *lines*, of dimension 2 are called *planes*.

Problem 11.2: Prove that the intersection of finitely many linear subspaces is again a linear subspace.

Solution 11.2:

Problem 11.3: Consider the affine space $\mathbb{C}^4$ with variables x, y, z, and w. Consider the planes P_1 and P_2 defined by $\langle x+y+z, x+y+w\rangle$ and $\langle x-y-z, x-y-w\rangle$, respectively. Find the ideal defining the linear space $P_1 \cap P_2$.

Solution 11.3:

Problem 11.4: In $\mathbb{C}^3$, find the ideal of the linear subspace perpendicular to $(1, 1, 1)$ and $(1, 1, -1)$ that passes through the point $(0, 1, 0)$.

Solution 11.4:

The are essentially two ways to describe a variety in $\mathbb{C}^n$. One way is to describe its ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$. A second possible way is to try to parametrize it. For instance, in $\mathbb{C}^2$ we have the line given by the ideal $\langle x - y \rangle$. This line can also be written as

$$\{(t, t) \mid t \in \mathbb{C}\}.$$

The main issue with parametrizations, is that sometimes the equations that we need to parametrize an affine variety are very complicated (much worse than polynomial equations). For example, you can try to parametrize the variety

$$\{(x, y) \mid y^2 = x^3 + x + 1\} \subset \mathbb{R}^2$$

with polynomial equations.

Thus, working with ideals gives us the advantage that we always stay within the world of polynomial equations.

Problem 11.5: Given the parametrization of the variety find its ideal.

- $\{(t, t + 1) \mid t \in \mathbb{C}\} \subset \mathbb{C}^2$.
- $\{(t, t + 1, 2t - 2) \mid t \in \mathbb{C}\} \subset \mathbb{C}^3$.
- $\{(t, t^2) \mid t \in \mathbb{C}\} \subset \mathbb{C}^2$.

Solution 11.5:

Problem 11.6: Describe the ideal of the following parametrized curve in $\mathbb{C}^3$:

$$\{(t, t^3, t^5) \mid t \in \mathbb{C}\} \subset \mathbb{C}^3.$$

Solution 11.6:

The *affine rational normal curve of degree* n is the curve in $\mathbb{C}^n$ parametrized by

$$\{(t, t^2, t^3, \ldots, t^n) \mid t \in \mathbb{C}\}.$$

Problem 11.7: Describe the ideal of the affine rational normal curve of degree n.

What is the minimum number of generators for this ideal?

Solution 11.7:

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