OLGA RADKO MATH CIRCLE: ADVANCED 3

JOAQUÍN MORAGA

Worksheet 9: Affine Varieties

Let F be a field. An *affine variety* is a subset $V \subseteq F^n$ defined by a finite set of polynomials, i.e., there exists $p_1, \dots, p_r \in F[x_1, \dots, x_n]$ for which

$$V = \{(f_1, \dots, f_n) \mid p_1(F_1, \dots, f_n) = p_r(f_1, \dots, f_n) = 0\}.$$

All the geometric objects that we have been working with are affine varieties.

The word *variety* essentially means geometric object that can be written using polynomials. On the other hand, the word *affine* means that it lives in the space F^n. The space F^n, also denoted as $\mathbb{A}^n_F$ is called the *n-dimensional affine space* over F.

Problem 9.0: Show that the following geometric objects are affine varieties:

- A finite set of points in $\mathbb{R}$.
- A finite set of points in $\mathbb{C}$.
- A line in $\mathbb{R}^2$.
- An ellipsoid in $\mathbb{R}^2$.
- A hyperbola in $\mathbb{R}^2$.
- The empty set and F^n.

Solution 9.0:

Although a lot of geometric objects in F^n are affine varieties, there are some geometric objects that are not.

Problem 9.1: Show that the following geometric objects are not affine varieties:

- The set $\mathbb{Z} \subset \mathbb{R}$.
- The set $\mathbb{Z}^2 \subset \mathbb{R}^2$, i.e., all the points $(x, y) \in \mathbb{R}^2$ wit both x and y integers.
- The graph of the sin function in $\mathbb{R}^2$.
- The graph of the cos function in $\mathbb{R}^2$.
- An infinite set of points in $\mathbb{R}^2$.

Solution 9.1:

Let $V \subset F^n$ be an affine variety. We say that V is *prime* (or *irreducible*) if whenever we write

$$V = W_1 \cup W_2,$$

where W_1 and W_2 are affine varieties, then either $W_1 = V$ or $W_2 = V$.

Let $V \subset F^n$ be an affine variety. Assume that we can write

$$V = W_1 \cup \cdots \cup W_2,$$

where each W_i is a prime affine variety. Then, we say that the W_i's are the *prime components* (or *irreducible components*) of V.

The previous definition should remind you of the definition of prime number and prime polynomial. Is there a relation between these objects?

Problem 9.2: Consider the following affine varieties. Decide whether the affine variety is prime or not. If not, find its prime components.

- The union of three lines in $\mathbb{R}^2$.
- Two circles in $\mathbb{R}^2$ that are tangent at the origin.
- Two ellipsoids in $\mathbb{R}^2$; one containing the other.
- A finite set of points in $\mathbb{R}^3$.

Solution 9.2:

Let $C \subset \mathbb{R}^2$ be an affine curve, i.e., the vanishing locus of a polynomial $p[x,y] \in \mathbb{R}[x,y]$. We say that C is *connected* if for every two points $p_1, p_2 \in C$ we can draw a continuous line from p_1 to p_2 from within the line, i.e., not touching $\mathbb{R}^2 \setminus C$.

Problem 9.3: Let $C \subset \mathbb{R}^2$ be a prime affine curve. Is it true that C is connected? If this is not true, show a counter-example.

Solution 9.3:

Let V be an affine variety in F^n. We consider the set of polynomials:

$$\{p \in F[x_1, \ldots, x_n] \mid p(x) = 0 \text{ for all } x \in V\}.$$

In other words, we are considering the set of all the polynomials that vanish at V. This set will be denoted by $I(V) \subset F[x_1, \ldots, x_n]$.

Problem 9.4: Let $V \subset F^n$ be an affine variety. Show that $I(V)$ is an ideal.
Solution 9.4:

Let R be a ring and I an ideal in R. We say that the elements $i_1, \dots, i_k \in I$ *generate the ideal* if every element $f \in I$ can be written as

$$f = f_1 i_1 + \cdots + f_k i_k,$$

for certain elements $f_i \in R$. If the elements $i_1, \dots, i_k$ generate the ideal I, then we write the equality

$$I = \langle i_1, \dots, i_k \rangle.$$

For an affine variety $V \subset F^n$, the ideal $I(V)$ is called the *ideal associated to the affine variety.*

Problem 9.5: For the following affine varieties $V \subset F^n$, write down a set of generators of the ideal $I(V)$.

- A point in $\mathbb{R}^n$.
- A line in $\mathbb{C}^3$.
- The union of the coordinate axes in $\mathbb{R}^2$.
- The union of a circle of radius one centered at the origin and a circle of radius two centered at the origin. Both of them in $\mathbb{R}^2$.

Solution 9.5:

Recall that given an ideal $I \subset F[x_1, \ldots, x_n]$, we denote by $V(I)$ the affine variety that it defines.

Problem 9.6: Let V_1 and V_2 be two affine varieties in the affine space F^n. Show that $V_1 \supseteq V_2$ if and only if $I(V_1) \subseteq I(V_2)$.

Let V be an affine variety in F^n. Show that

$$V(I(V)) = V.$$

In other words, if we *transform* the variety into an ideal and then we *transform* the ideal into a variety, we go back to the initial variety.

Solution 9.6:

In the previous problem we showed that

$$V(I(V)) = V.$$

Does the same happens if we start with an ideal? We will explore this question in the following problem.

Problem 9.7: Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$. Show that

$$I(V(I)) \supseteq I.$$

Can you find an example in which the equality does not hold?

Solution 9.7:

Problem 9.8: Show that the equality

$$I(V(I)) = \sqrt{I}$$

holds for the following ideals.

- The ideal generated by $(x^2 + 1)^2$.
- The ideal generated by x^2, y^3, and z^4.
- The ideal generated by $x^2 + y^2 - 1$.

Solution 9.8:

The following theorem is known as Hillbert's Nullstellensatz.

Theorem 1. *Let I be an ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then, we have that $\sqrt{I} = I(V(I))$.*

Problem 9.9: Show that the Hillbert's Nullstellensatz is not valid over the real numbers.

Deduce the following statement from Hillbert's Nullstellensatz: If $f_1, \dots, f_r$ are complex polynomials, and we can find complex polynomials $g_1, \dots, g_r$ such that

$$f_1 g_1 + \cdots + f_r g_r = 1,$$

then the polynomials f_i do not have a common zero.

Solution 9.9:

Let R be a ring and I be an ideal. We say that I is a *prime* ideal if $fg \in I$ implies that either f is in I or g is in I.

Problem 9.10: Let I be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Show that I is prime if and only if $V(I)$ is prime.

Solution 9.10:

Problem 9.11: Determine whether the following ideals are prime or not:

- The ideal generated by the monomial xy in $\mathbb{R}[x, y]$.
- The ideal generated by the monomials x^2y, xy^2 in $\mathbb{C}[x, y]$.
- The ideal generated by $(x^2 + 1)^2$ in $\mathbb{R}[x, y]$.
- The ideal generated by $x^2 + y^2 - 1$ in $\mathbb{C}[x, y]$.

Solution 9.11:

In summary, we have a *duality* between algebra and geometry. If we want to understand the algebra of an ideal I, we can use the information of the geometric object $V(I)$ to deduce data about I. On the other way around, if we want to understand geometric data about an affine variety V, we can use the algebraic data $I(V)$. This gives us a *bridge* between two worlds: *The Algebraic World* and *The Geometric World*.

We have the following correspondences:

$$\begin{array}{ccc}
\{\text{Radical Ideals in } \mathbb{C}[x_1, \ldots, x_n]\} & \underset{I}{\overset{V}{\longleftrightarrow}} & \{\text{Affine varieties in } \mathbb{C}^n\} \\
\uparrow & & \uparrow \\
\{\text{Prime ideals in } \mathbb{C}[x_1, \ldots, x_n]\} & \underset{I}{\overset{V}{\longleftrightarrow}} & \{\text{Prime affine varieties in } \mathbb{C}^n\}
\end{array}$$

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: jmoraga@math.ucla.edu