OLGA RADKO MATH CIRCLE: ADVANCED 3

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Worksheet 8: Vanishing Sets and Ideals

Let $\mathcal{S} := \{p_1, \dots, p_k\}$ be a finite set of polynomials in $F[x_1, \dots, x_r]$. The *vanishing set* (or *vanishing locus*) of $\mathcal{S}$ is the subset

$$\{(f_1, \dots, f_r) \in F^r \mid p_1(f_1, \dots, f_r) = \dots = p_k(f_1, \dots, f_r) = 0\}.$$

In other words, the vanishing set is the set of all the points in F^r at which all the polynomials of the finite set vanish.

We will start by computing the vanishing locus of a set of two polynomials in two variables.

Problem 8.0: Consider $\mathbb{R}$ the field of real numbers. Let $\mathbb{R}[x, y]$ be the ring of real polynomials in two variables. For the following sets of two polynomials represent their vanishing loci in $\mathbb{R}^2$. Also represent the vanishing locus of each polynomial in the set.

- $\mathcal{S} := \{x + y, x - y\}$.
- $\mathcal{S} := \{x^2 + y^2 - 4, x + y\}$.
- $\mathcal{S} := \{x^2 + 4y^2 - 4, 4x^2 + y^2 - 4\}$.

Solution 8.0:

For a polynomial p in $F[x_1, \ldots, x_r]$, we denote its vanishing locus in F^r by $V(p)$. For a set of polynomials $\mathcal{S}$, we denote the vanishing locus in F^r by $V(\mathcal{S})$.

Problem 8.1: Let $f_1, \ldots, f_k$ be k polynomials in the polynomial ring $F[x_1, \ldots, x_r]$. Let $\mathcal{S} := \{f_1, \ldots, f_k\}$. Show that the following equality of sets holds:

$$V(\mathcal{S}) = V(f_1) \cap \cdots \cap V(f_k).$$

Solution 8.1:

Recall that a polynomial p is called *irreducible* if whenever we can write

$$p = q_1 q_2,$$

for two polynomials q_i, then either q_1 or q_2 is a constant. Irreducible polynomials are also called *prime* as they play a role similar to such of prime numbers.

For instance, the polynomial $x^2 + y^2$ is irreducible in $\mathbb{R}[x, y]$.

Let p be a polynomial in $F[x_1, \ldots, x_r]$. The expression

$$p = q_1 \cdot \cdots \cdot q_s,$$

is called a *factorization into prime factors* if each q_i is prime (irreducible).

Problem 8.2: For the following polynomials, write a factorization into prime factors.

- The polynomial $x^3 + x^2y + xy^2 - 4x + y^3 - 4y$.
- The polynomial $x^4 + 2x^2y^2 - 10x^2 + y^4 - 10y^2 + 24$.
- The polynomial $4x^4 + 17x^2y^2 - 20x^2 + y^4 - 20y^2 + 16$.

Solution 8.2:

Problem 8.3: Let p be a polynomial and

$$p = q_1 \cdot \dots \cdot q_k$$

be a factorization into prime factors. Prove that

$$V(p) = V(q_1) \cup \cdots \cup V(q_k).$$

Then, verify this formula for the prime factorizations that you found in Problem 4.2.

Solution 8.3:

Let $x_1^{a_1} \cdot \dots \cdot x_r^{a_r}$ be a monomial in $F[x_1, \dots, x_r]$. The *degree* of this monomial is $a_1 + \dots + a_r$. Let $p \in F[x_1, \dots, x_r]$. Recall that the *degree* of p is the largest degree among the monomials that appear in p.

For example, the polynomial

$$p = x^3 + xy + y^4$$

has degree 4 as this is the largest positive integer among the numbers $\{3, 1+1, 4\}$.

Problem 8.4: Let p be a polynomial of degree d and let l be a linear polynomial, i.e., a polynomial of degree one. Both polynomials in $\mathbb{R}[x, y]$.

Show that $V(p) \cap V(l)$ consists of at most d points.

Try to find a "geometric explanation" for this phenomena.

Solution 8.4:

Problem 8.5: For which positive integers d can you find a prime polynomial of degree d in $\mathbb{C}[x]$?

For which positive integers d can you find a prime polynomial of degree d in $\mathbb{R}[x]$?

For which positive integers d can you find a prime polynomial of degree d in $\mathbb{R}[x, y]$? Are these polynomials still prime in $\mathbb{C}[x, y]$?

Solution 8.5:

In the following problem, we will describe the vanishing locus of a set of two polynomials in $\mathbb{R}[x, y, z]$.

Problem 8.6: For the following sets of two polynomials in $\mathbb{R}[x, y, z]$ find the vanishing set of of each polynomial and the vanishing locus of the set itself.

- $\mathcal{S} := \{x^2 + y^2 - 4, x^2 + y^2 + z^2 - 5\}$.
- $\mathcal{S} := \{x^2 - y^2 + z^2 - 1, z - x^2 - y^2\}$.

Solution 8.6:

Problem 8.7: Let p_1 and p_2 be two polynomials in $F[x_1, \dots, x_r]$. Show that the following containments hold:

- $V(\lambda p_1) = V(p_1)$ for each $\lambda \in F$ different from zero.
- $V(p_1 + p_2) \supset V(p_1) \cap V(p_2)$.
- $V(p_1 p_2) \supset V(p_1) \cap V(p_2)$.
- $V(p_1^r) = V(p_1)$.

Solution 8.7:

Let R be a ring. An *ideal* I of R is a subset satisfying the following conditions:

- $0 \in I$,
- If $r \in R$ and $i \in I$, then $ri \in I$, and
- If $i_1, i_2 \in I$, then $i_1 + i_2 \in I$.

Let $f_1, \ldots, f_s \in R$. The *ideal generated by* $f_1, \ldots, f_s$, denoted by $\langle f_1, \ldots, f_s \rangle$, is the set

$$\{f_1 g_1 + \cdots + f_s g_s \mid g_i \in R\}.$$

Problem 8.8: Let $f_1, \ldots, f_k \in R$ be elements of a ring. Show that $\langle f_1, \ldots, f_k \rangle$ is indeed an ideal.

Let $\mathbb{Z}$ be the ring of integers. Describe the ideals in $\mathbb{Z}$.

Let F be a field. Describe the ideals in F.

Let $F[x]$ be the polynomial ring over an algebraically closed field F. Describe the ideals of $F[x]$.

Solution 8.8:

Let I be an ideal in $F[x_1, \ldots, x_r]$. The *vanishing set* of an ideal is

$$\{(f_1, \ldots, f_r) \mid p(f_1, \ldots, f_r) = 0 \text{ for all } p \in I\} \subset F^r.$$

This set is denoted by $V(I)$.

The *radical* of an ideal $I \subset F[x_1, \ldots, x_r]$ consists of all elements $x \in F[x_1, \ldots, x_r]$ for which $x^m \in I$ for some power m. The radical of an ideal I is denoted by $\sqrt{I}$.

Problem 8.9: Show that the radical of an ideal is again an ideal.

Show that $V(I) = V(\sqrt{I})$.

Compute the radical of the ideal $\langle x^3, y^4, z^5 \rangle \subset \mathbb{C}[x, y, z]$ and compute its vanishing set.

Solution 8.9:

Problem 8.10: Let I_1 and I_2 be two ideals in $\mathbb{C}[x_1, \ldots, x_r]$. Show that the inequality $\sqrt{I_1} \subset \sqrt{I_2}$ holds if and only if $V(I_1) \supset V(I_2)$.

Let p_1 and p_2 be two prime polynomials. Show that if $V(p_2) \subset V(p_1)$, then $p_1 = p_2$.

Solution 8.10:

Problem 8.11: Let p_1 be a polynomial of degree d_1 in $\mathbb{R}[x, y]$. Let p_2 be a polynomial of degree d_2 in $\mathbb{R}[x, y]$. Show that $V(p_1) \cap V(p_2)$ consists of at most $d_1 d_2$ points.

Solution 8.11:

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