Worksheet 7: Polynomial Rings

In the previous quarter we learnt about rings. In this quarter, we will mostly work with rings of polynomials. Given a ring $R$, we denote by $R[x]$ the polynomial ring over $R$. The elements of $R[x]$ are polynomials whose coefficients belong to $R$ with the usual addition and multiplication of polynomials.

The roots of a polynomial $p(x) \in R[x]$ are all the elements $r \in R$ for which $p(r) = 0$.

**Problem 7.0:** Given a field $F$ and a polynomial $p(x)$ in $F[x]$ find all its roots.
- The field $\mathbb{R}$ and the polynomial $x^4 + 7x^2 + 12$.
- the field $\mathbb{C}$ and the polynomial $x^3 + x^2 + x + 1$.
- The field $\mathbb{Q}$ and the polynomial $3x^4 + 15x^2 + 10$.

**Solution 7.0:**
The ring $\mathbb{R}[x]$ is called the ring of real polynomials in one variable. The ring $\mathbb{C}[x]$ is called the ring of complex polynomials in one variable.

**Problem 7.1:** For the following polynomials find their real roots and complex roots. Then, represent the solution in the real line and in the complex plane.

- The polynomial $x^6 + x^3 + 1$.
- The polynomial $x^8 + 1$.
- The polynomial $x^4 + x^2 + 1$.

**Solution 7.1:**
Recall that given a prime number $p$, we denote by $\mathbb{Z}_p$ the field consisting of the $p$ elements $\{0, \ldots, p - 1\}$.

**Problem 7.2:** For the following polynomials with $\mathbb{Z}$-coefficients find all its roots in $\mathbb{Z}_p$ and represent them in the set $\{0, \ldots, p - 1\}$.

- The polynomial $x^4 + x^3 + x^2 + x + 1$ over the field $\mathbb{Z}_5$.
- The polynomial $x^{12} - x^2 + 1$ over the field $\mathbb{Z}_{13}$.
- The polynomial $x^{16} + x^3 + 1$ over the field $\mathbb{Z}_{17}$.

**Solution 7.2:**
Given a field $F$, we can consider the ring $R = F[x]$ of polynomials over $x$. Then, we may consider the ring of polynomials over $R$, denoted by $R[y]$. Note that we use the variable $y$ to avoid using the variable $x$ twice. The ring $R[y]$ is usually denoted by $F[x,y]$ and called the ring of polynomials over $F$ in two variables $x$ and $y$.

**Problem 7.3:** In this problem, we will consider two polynomials $p(x, y)$ and $q(x, y)$ in $F[x, y]$. We will compute their addition $p(x, y) + q(x, y)$ and their product $p(x, y) \times q(x, y)$. Then, we will write the outcome as a polynomial over $x$ with coefficients in $R[y]$ and as a polynomial over $y$ with coefficients in $R[x]$.

- The polynomial $p(x, y) = x^2 + y^2$ and the polynomial $q(x, y) = x^2 - y^2$.
- The polynomial $p(x, y) = x - y$ and the polynomial $q(x, y) = x^3 + y^3$.
- The polynomial $p(x, y) = xy$ and $q(x, y) = x^2 + xy + y^2$.

**Solution 7.3:**
We can iterate the construction that we introduced in the previous problem. Let $R = F[x_1, \ldots, x_n]$ be the polynomial ring over $F$ with variables $x_1, \ldots, x_n$. We can set a new variable $x_{n+1}$ and consider the ring $R[x_{n+1}]$. Then, the ring $R[x_{n+1}]$, usually denoted by $F[x_1, \ldots, x_{n+1}]$, is called the ring of polynomials over $F$ in $n + 1$ variables. The name of the variables usually does not matter. Changing their labels only change how the ring looks, however it does not change its nature.

**Problem 7.4:** In this problem, we will consider two polynomials $p(x, y, z)$ and $q(x, y, z)$ in $R[x, y, z]$. We will compute their addition $p(x, y, z) + q(x, y, z)$ and their product $p(x, y, z) \times q(x, y, z)$. Then, we will write the outcome as a polynomial over $z$ with coefficients in $R[x, y]$.

- The polynomial $p(x, y) = x^2 + y^2 + z^2$ and the polynomial $q(x, y) = x^2 - y^2 + z^2$.
- The polynomial $p(x, y) = x - y + z$ and the polynomial $q(x, y) = x^3 + y^3 + z^3$.
- The polynomial $p(x, y) = xyz$ and $q(x, y) = x + y + z$.

**Solution 7.4:**
At this point you may be wondering: Why do we care about these polynomial rings?

Let's explore one of the many reasons. Imagine you have a figure: for instance a sphere of radius 1 in the 3-dimensional space. If you want to explain this object to a computer, then you would have to input on the computer using algebraic equations, in other words, using equations. In this case, you are very likely to use the equation $x^2 + y^2 + z^2 = 1$.

Let $p = p(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. The vanishing set of $p$ is the set of all the points $(f_1, \ldots, f_n) \in F^n$ with $p(f_1, \ldots, f_n) = 0$. The vanishing set is also called the vanishing locus. If $n = 1$, then the vanishing set is nothing else than the set of roots in $F$. The vanishing set of a polynomial $p$ is often denoted by $V(p)$.

Example: Consider the polynomial $p(x, y) = x^2 + y^2 - 1$ in $\mathbb{R}[x, y]$. Then the vanishing set of $p$ is simply a circle of radius 1 centered at the origin in $\mathbb{R}^2$.

In the following problem, you will try to draw the vanishing set of a polynomial in two variables over $\mathbb{R}$ as precisely as possible.

Problem 7.5:
- Draw the vanishing set of the polynomial $x^2 - xy + y^2 - 5$ in $\mathbb{R}^2$.
- Draw the vanishing set of the polynomial $x^2 + xy + y^2 - 5$ in $\mathbb{R}^2$.

Solution 7.5:
The following are more sophisticated examples.

**Problem 7.6:** Try to use your calculator and draw as well as possible the vanishing locus of the following polynomials:
- the polynomial $x^3 + y^3 - 2xy$, and
- the polynomial $x^4 + y^4 - x^2 - y^2 - 100$.

**Solution 7.6:**
The following is a slightly more abstract example as the set $F^2$ is discrete this time.

**Problem 7.7:** Find the vanishing locus of $x^{10} + y^{10} + xy$ in $\mathbb{Z}_3^2$.
Find the vanishing locus of $x^5 + y^5 + x^3 + y^3 - 2xy$ in $\mathbb{Z}_5^2$.

**Solution 7.7:**
Consider the square $S$ in $\mathbb{R}^2$ with vertices $(1, 1), (-1, 1), (1, -1), \text{ and } (-1, -1)$.

**Problem 7.8:** Is it possible to find a polynomial $p(x, y) \in \mathbb{R}[x, y]$ whose vanishing locus is exactly $S$? If yes, explain how. If no, explain why.

**Solution 7.8:**
In the following example, we will play with the polynomial ring \( \mathbb{R}[x,y,t] \). In this case, instead of considering the variable \( t \) as a \textit{coordinate variable}, we will consider it as a \textit{time variable}. In other words, the polynomial \( p(x, y, t) = xy + t \) equals \( xy \) at the time 0, equals \( xy + 1 \) at the time 1, and so on.

**Problem 7.9:** Consider the polynomial 

\[
p(x, y, t) = t - x^2 + y^2
\]

in the ring \( \mathbb{R}[x,y,t] \). For the times \( t \in \{-3, -1, 0, 1, 3\} \), draw the vanishing set of \( p(x, y, t) \). Draw all the vanishing sets in the same space \( \mathbb{R}^2 \), i.e., in the same Euclidean space.

Can you describe what is happening to the vanishing set when the time goes to \( \infty \)?

**Solution 7.9:**
The following example shows that many variables as \textit{time}, \textit{temperature}, etc. can be viewed as coordinates in a higher-dimensional space.

**Problem 7.10:** Consider the polynomial \( p(x, y, z) = z - x^2 + y^2 \) in \( \mathbb{R}[x, y, z] \). Describe the vanishing set of this polynomial in \( \mathbb{R}^3 \).

Explain how the solution of this problem relates to the solution that you found in Problem 4.9.

**Solution 7.10:**
The vanishing set of a polynomial \( p(x, y) \in \mathbb{R}[x, y] \) in \( \mathbb{R}^2 \) is called a real algebraic curve. The degree of a real algebraic curve is the maximum exponent \( a + b \) among the monomials \( x^a y^b \) appearing in \( p \). A curve of degree one is called a line, a curve of degree two is called a conic, and a curve of degree three is called a cubic.

**Problem 7.11:** Consider 2 points in \( \mathbb{R}^2 \). Can you find a line passing through those two points?

Consider 3 points in \( \mathbb{R}^2 \). Can you find a conic passing through those three points?

Find the maximum integer \( n \) so that for every \( n \) points in \( \mathbb{R}^2 \) we can find a conic passing through them.

**Solution 7.11:**