OLGA RADKO MATH CIRCLE: ADVANCED 3

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Worksheet 7: Polynomial Rings

In the previous quarter we learnt about rings. In this quarter, we will mostly work with rings of polynomials. Given a ring R, we denote by $R[x]$ the polynomial ring over R. The elements of $R[x]$ are polynomials whose coefficients belong to R with the usual addition and multiplication of polynomials.

The *roots* of a polynomial $p(x) \in R[x]$ are all the elements $r \in R$ for which $p(r) = 0$.

Problem 7.0: Given a field F and a polynomial $p(x)$ in $F[x]$ find all its roots.

- The field $\mathbb{R}$ and the polynomial $x^4 + 7x^2 + 12$.
- the field $\mathbb{C}$ and the polynomial $x^3 + x^2 + x + 1$.
- The field $\mathbb{Q}$ and the polynomial $3x^4 + 15x^2 + 10$.

Solution 7.0:

The ring $\mathbb{R}[x]$ is called the ring of real polynomials in one variable.
The ring $\mathbb{C}[x]$ is called the ring of complex polynomials in one variable.

Problem 7.1: For the following polynomials find their real roots and complex roots. Then, represent the solution in the real line and in the complex plane.

- The polynomial $x^6 + x^3 + 1$.
- The polynomial $x^8 + 1$.
- The polynomial $x^4 + x^2 + 1$.

Solution 7.1:

Recall that given a prime number p, we denote by $\mathbb{Z}_p$ the field consisting of the p elements $\{0, \ldots, p-1\}$.

Problem 7.2: For the following polynomials with $\mathbb{Z}$-coefficients find all its roots in $\mathbb{Z}_p$ and represent them in the set $\{0, \ldots, p-1\}$.

- The polynomial $x^4 + x^3 + x^2 + x + 1$ over the field $\mathbb{Z}_5$.
- The polynomial $x^{12} - x^2 + 1$ over the field $\mathbb{Z}_{13}$.
- The polynomial $x^{16} + x^3 + 1$ over the field $\mathbb{Z}_{17}$.

Solution 7.2:

Given a field F, we can consider the ring $R = F[x]$ of polynomials over x. Then, we may consider the ring of polynomials over R, denoted by $R[y]$. Note that we use the variable y to avoid using the variable x twice. The ring $R[y]$ is usually denoted by $F[x, y]$ and called the *ring of polynomials over F in two variables x and y*.

Problem 7.3: In this problem, we will consider two polynomials $p(x, y)$ and $q(x, y)$ in $F[x, y]$. We will compute their addition $p(x, y) + q(x, y)$ and their product $p(x, y) \times q(x, y)$. Then, we will write the outcome as a polynomial over x with coefficients in $\mathbb{R}[y]$ and as a polynomial over y with coefficients in $\mathbb{R}[x]$.

- The polynomial $p(x, y) = x^2 + y^2$ and the polynomial $q(x, y) = x^2 - y^2$.
- The polynomial $p(x, y) = x - y$ and the polynomial $q(x, y) = x^3 + y^3$.
- The polynomial $p(x, y) = xy$ and $q(x, y) = x^2 + xy + y^2$.

Solution 7.3:

We can iterate the construction that we introduced in the previous problem. Let $R = F[x_1, \ldots, x_n]$ be the polynomial ring over F with variables $x_1, \ldots, x_n$. We can set a new variable x_{n+1} and consider the ring $R[x_{n+1}]$. Then, the ring $R[x_{n+1}]$, usually denoted by $F[x_1, \ldots, x_{n+1}]$, is called the *ring of polynomials over F in $n+1$ variables*. The name of the variables usually does not matter. Changing their labels only change how the ring looks, however it does not change its *nature*.

Problem 7.4: In this problem, we will consider two polynomials $p(x, y, z)$ and $q(x, y, z)$ in $\mathbb{R}[x, y, z]$. We will compute their addition $p(x, y, z) + q(x, y, z)$ and their product $p(x, y, z) \times q(x, y, z)$. Then, we will write the outcome as a polynomial over z with coefficients in $\mathbb{R}[x, y]$.

- The polynomial $p(x, y) = x^2 + y^2 + z^2$ and the polynomial $q(x, y) = x^2 - y^2 + z^2$.
- The polynomial $p(x, y) = x - y + z$ and the polynomial $q(x, y) = x^3 + y^3 + z^3$.
- The polynomial $p(x, y) = xyz$ and $q(x, y) = x + y + z$.

Solution 7.4:

At this point you may be wondering: Why do we care about these polynomial rings?

Lets explore one of the many reasons. Imagine you have a figure: for instance a sphere of radius 1 in the 3-dimensional space. If you want to explain this object to a computer, then you would have to input on the computer using algebraic equations, in other words, using equations. In this case, you are very likely to use the equation $x^2 + y^2 + z^2 = 1$.

Let $p = p(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. The *vanishing set* of p is the set of all the points $(f_1, \ldots, f_n) \in F^n$ with $p(f_1, \ldots, f_n) = 0$. The vanishing set is also called the *vanishing locus*. If $n = 1$, then the vanishing set is nothing else than the set of roots in F. The vanishing set of a polynomial p is often denoted by $V(p)$.

Example: Consider the polynomial $p(x, y) = x^2 + y^2 - 1$ in $\mathbb{R}[x, y]$. Then the vanishing set of p is simply a circle of radius 1 centered at the origin in $\mathbb{R}^2$.

In the following problem, you will try to draw the vanishing set of a polynomial in two variables over $\mathbb{R}$ as precisely as possible.

Problem 7.5:

- Draw the vanishing set of the polynomial $x^2 - xy + y^2 - 5$ in $\mathbb{R}^2$.
- Draw the vanishing set of the polynomial $x^2 + xy + y^2 - 5$ in $\mathbb{R}^2$.

Solution 7.5:

The following are more sophisticated examples.

Problem 7.6: Try to use your calculator and draw as well as possible the vanishing locus of the following polynomials:

- the polynomial $x^3 + y^3 - 2xy$, and
- the polynomial $x^4 + y^4 - x^2 - y^2 - 100$.

Solution 7.6:

The following is a slightly more abstract example as the set F^2 is discrete this time.

Problem 7.7: Find the vanishing locus of $x^{10} + y^{10} + xy$ in $\mathbb{Z}_3^2$.
Find the vanishing locus of $x^5 + y^5 + x^3 + y^3 - 2xy$ in $\mathbb{Z}_5^2$.

Solution 7.7:

Consider the square S in $\mathbb{R}^2$ with vertices $(1,1), (-1,1), (1,-1)$, and $(-1,-1)$.

Problem 7.8: Is it possible to find a polynomial $p(x,y) \in \mathbb{R}[x,y]$ whose vanishing locus is exactly S? If yes, explain how. If no, explain why.

Solution 7.8:

In the following example, we will play with the polynomial ring $\mathbb{R}[x, y, t]$. In this case, instead of considering the variable t as a *coordinate variable*, we wil consider it as a *time variable*. In other words, the polynomial $p(x, y, t) = xy + t$ equals xy at the time 0, equals $xy + 1$ at the time 1, and so on.

Problem 7.9: Consider the polynomial

$$p(x, y, t) = t - x^2 + y^2$$

in the ring $\mathbb{R}[x, y, t]$. For the times $t \in \{-3, -1, 0, 1, 3\}$, draw the vanishing set of $p(x, y, t)$. Draw all the vanishing sets in the same space $\mathbb{R}^2$, i.e., in the same Euclidean space.

Can you describe what is happening to the vanishing set when the time goes to ∞?

Solution 7.9:

The following example shows that many variables as *time*, *temperature*, etc. can be viewed as coordinates in a *higher-dimensional space*.

Problem 7.10: Consider the polynomial $p(x, y, z) = z - x^2 + y^2$ in $\mathbb{R}[x, y, z]$. Describe the vanishing set of this polynomial in $\mathbb{R}^3$.

Explain how the solution of this problem relates to the solution that you found in Problem 4.9.

Solution 7.10:

The vanishing set of a polynomial $p(x, y) \in \mathbb{R}[x, y]$ in $\mathbb{R}^2$ is called a *real algebraic curve*. The *degree* of a real algebraic curve is the maximum exponent $a + b$ among the monomials $x^a y^b$ appearing in p. A curve of degree one is called a *line*, a curve of degree two is called a *conic*, and a curve of degree three is called a *cubic*.

Problem 7.11: Consider 2 points in $\mathbb{R}^2$. Can you find a line passing through those two points?

Consider 3 points in $\mathbb{R}^2$. Can you find a conic passing through those three points?

Find the maximum integer n so that for every n points in $\mathbb{R}^2$ we can find a conic passing through them.

Solution 7.11:

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