Vector Spaces Over Fields

Up to this point, we have investigated the algebraic properties of numbers. In particular, number systems that have all of the properties we like are called fields. One way we can use fields to model the real world is to use *vectors*.

The arrow and coordinate representation of vectors

In your life-long journey studying the beauty of mathematics, you will have to reference physical or online resources such as textbooks and videos. One of our favorite math YouTubers is the channel 3Blue1Brown by Grant Sanderson. We will spend the beginning of this lesson watching his video:

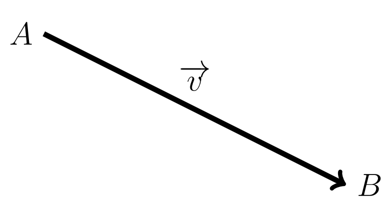
Vectors | Chapter 1, Essence of linear algebra
`https://www.youtube.com/watch?v=fNk_zzaMoSs`

We will watch this video in parts, supplementing it with some computational problems. As Grant tends to say, his videos are best understood when you "pause and ponder" often.

Watch 0:00 to 1:25 for vector interpretations

We learned the physics perspective of vectors last school year and used it to prove some geometry facts! Let's see if we can jog our memory.

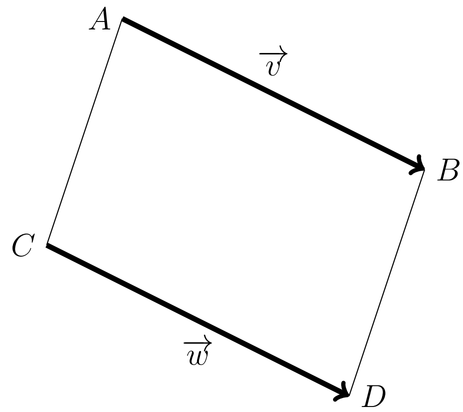
Definition 1. *A vector in the Euclidean plane is a directed segment or arrow. In particular, a vector has a length and a direction.*



$$\vec{v} = \overrightarrow{AB}$$

For the vector $\vec{v} = \overrightarrow{AB}$, point A is called the initial point and point B is called the terminal point.

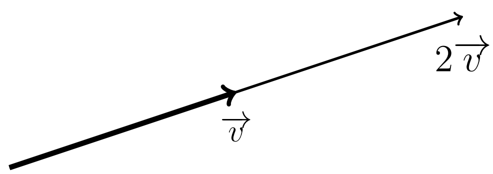
Definition 2. *Two vectors,* $\overrightarrow{v} = \overrightarrow{AB}$ *and* $\overrightarrow{w} = \overrightarrow{CD}$ *are defined to be equal if the quadrilateral* $ABDC$ *is a parallelogram.*



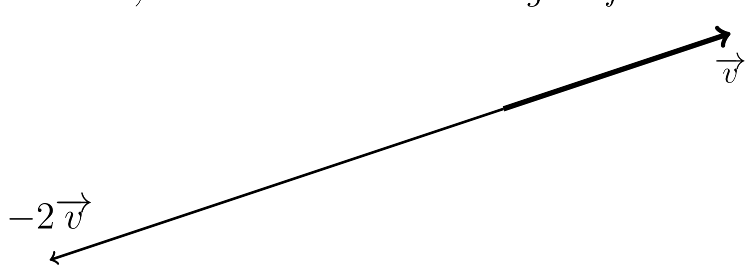
In other words, two vectors $\overrightarrow{v}, \overrightarrow{w}$ *are considered the same if you can move* $\overrightarrow{v}$ *so that its initial and terminal points lie on top of the initial and terminal points of* $\overrightarrow{w}$*. This shows that the vectors have the same direction and length, even though they start at different points. In this case, we write* $\overrightarrow{v} = \overrightarrow{w}$*.*

Definition 3. *A vector whose initial and terminal points coincide is called the zero vector and is written* $\overrightarrow{0}$*. The zero vector has zero length and hence points in no direction.*

Definition 4. *Let* $\overrightarrow{v}$ *be a vector in the Euclidean plane and let* $t > 0$ *be a real number. Then we can define* $t\overrightarrow{v}$ *with the same initial point as* $\overrightarrow{v}$*, points in the same direction as* $\overrightarrow{v}$*, but has* t *times the length of* v*. For example:*

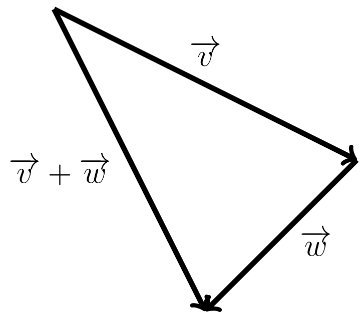


For $t < 0$*, we can define* $t\overrightarrow{v}$ *with the same initial point as* $\overrightarrow{v}$*, points in the opposite direction as* $\overrightarrow{v}$*, but has* t *times the length of* v*. For example:*



Finally, we define $0\overrightarrow{v} = \overrightarrow{0}$ *for any vector* $\overrightarrow{v}$*.*

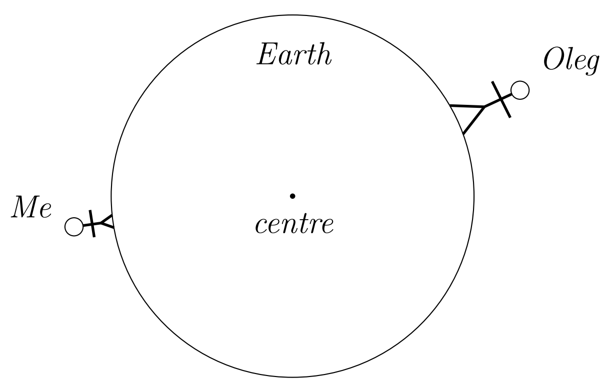
Definition 5. *Let $\overrightarrow{v}, \overrightarrow{w}$ be vectors in the Euclidean plane. We define the addition of $\overrightarrow{v}$ and $\overrightarrow{w}$ as follows. Move $\overrightarrow{w}$ so that the initial point of $\overrightarrow{w}$ coincides with the terminal point of $\overrightarrow{v}$. The vector originating at the initial point of $\overrightarrow{v}$ and terminating at the terminal point of $\overrightarrow{w}$ is the sum $\overrightarrow{v} + \overrightarrow{w}$.*



Okay, that's a lot of definitions although they are hopefully familiar and geometrically obvious to you. Before moving on with the video, let's recall two usages of vectors in physics.

Physical forces, such as the force of gravity or the electric force that pulls together two objects having a different electric charge and pushes away two objects having the same electric charge, are vectors in 3D. The direction of a vector shows the direction in which the corresponding force is acting. The length of the vector shows the strength of the force.

Problem 1. *On the picture below, draw the vectors of the gravitational pull the Earth exerts on you and on your Math Circle leader.*

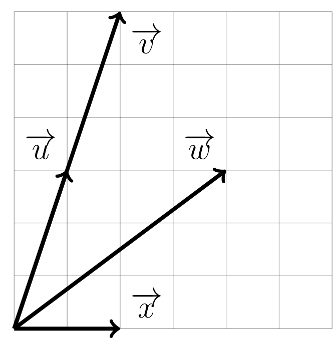


Problem 2. *Assuming that Oleg is twice as heavy as you are, how do you show it by means of the gravitational pull vectors on the above picture?*

Watch 1:25 to 4:36 for vector coordinates

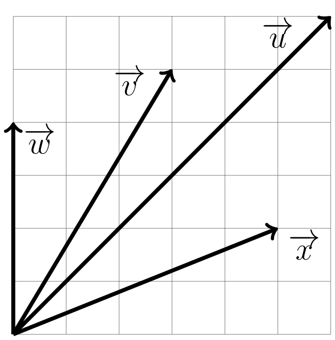
The video has identified the first very important tool for us: identifying the vector arrow definition with the coordinate definition. We should emphasize that, if you want to model a vector with numbers, then you have to make sure that every vector you use starts at the same point: origin.

Problem 3. *Fill in the coordinates of the vectors depicted on the plane below. Each grid line is one unit apart and the vectors all start at the origin.*



$$\overrightarrow{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{x} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Problem 4. *Fill in the coordinates of the vectors depicted on the plane below. Each grid line is one unit apart and the vectors all start at the origin.*

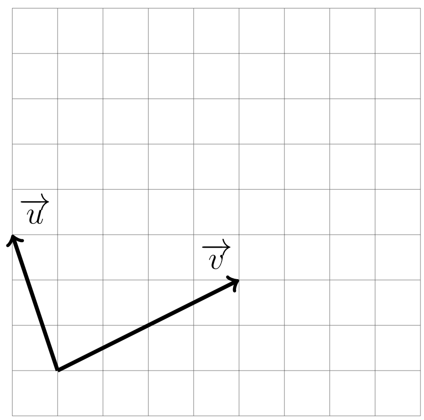


$$\overrightarrow{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{w} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{x} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Watch 4:36 to 8:10 for vector addition and scaling

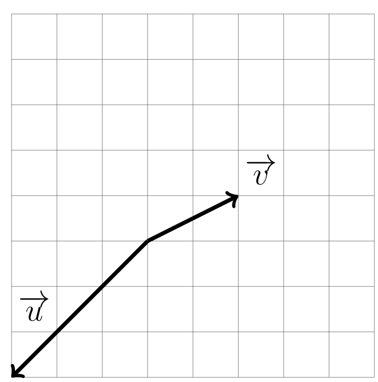
We can interpret vector addition in coordinates very easily!

Problem 5. *Fill in the coordinates of the vectors* $\vec{u}, \vec{v}$ *depicted on the plane below. Each grid line is one unit apart and the vectors all start at the origin. Then compute the coordinates of* $2\vec{u} + \vec{v}$ *and draw it on the grid.*



$$\vec{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad 2\vec{u} + \vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Problem 6. *Fill in the coordinates of the vectors* $\vec{u}, \vec{v}$ *depicted on the plane below. Each grid line is one unit apart and the vectors all start at the origin. Then compute the coordinates of* $\frac{-1}{3}\vec{u} + 2\vec{v}$ *and draw it on the grid.*



$$\vec{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \frac{-1}{3}\vec{u} + 2\vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

The 2-dimensional vector space over $\mathbb{R}$

Overall, the identification of vectors with coordinates allows us to easily construct the set of all vectors in the plane.

Definition 6. *A 2-dimensional real vector is a pair of two real numbers u_1, u_2 written as*

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We define the 2-dimensional vector space over $\mathbb{R}$ to be the set of all 2-dimensional real vectors, which we denote by $\mathbb{R}^2$. We define the binary operation of addition $+$ on $\mathbb{R}^2$ as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

We define the binary operation of scalar multiplication $\cdot$ on $\mathbb{R}$ and $\mathbb{R}^2$ as

$$t \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} tu_1 \\ tu_2 \end{bmatrix}.$$

In this context, we typically call the real number t a scalar. Note that we usually drop the symbol $\cdot$ and simply write

$$t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} tu_1 \\ tu_2 \end{bmatrix}$$

when the context is clear.

Remark 1. *When mathematicians say the "Euclidean plane", they are often actually referring to $\mathbb{R}^2$. This is because each vector in $\mathbb{R}^2$ identifies a unique point on the 2-dimensional plane, namely the terminal point of the arrow associated with the coordinate representation of the vector.*

We know from the YouTube video that Definition 6 can be identified with the physic's arrow definition of vectors. The benefit of using coordinates instead of arrows is that it lets us prove many facts that would be very difficult to prove geometrically with arrows. Indeed, last year we learned that $\mathbb{R}^2$ satisfies the following axioms which can be very easily proved using coordinates. Use the algebraic properties of $\mathbb{R}$ and don't over think these proofs.

Axiom 1 (Closure of vector addition)**.** *For vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $\mathbb{R}^2$*, we know that* $\overrightarrow{u} + \overrightarrow{v}$ *is also in* $\mathbb{R}^2$*.*

Problem 7. *Prove that vector addition in* $\mathbb{R}^2$ *is closed.*

Axiom 2 (Associativity of vector addition)**.** *For vectors* $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ *in* $\mathbb{R}^2$*, we know that* $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w}$*. This lets us ignore parentheses when adding.*

Problem 8. *Prove that vector addition in* $\mathbb{R}^2$ *is associative.*

Axiom 3 (Commutativity of vector addition)**.** *For vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $\mathbb{R}^2$*, we know that* $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$*.*

Problem 9. *Prove that vector addition in* $\mathbb{R}^2$ *is commutative.*

Axiom 4 (Identity of vector addition)**.** *There is a vector* $\overrightarrow{a}$ *in* $\mathbb{R}^2$ *such that, for all vectors* $\overrightarrow{u}$ *in* $\mathbb{R}^2$*, we have* $\overrightarrow{a} + \overrightarrow{u} = \overrightarrow{u} + \overrightarrow{a} = \overrightarrow{u}$*. We call* $\overrightarrow{a}$ *the additive identity of* $\mathbb{R}^2$*.*

Problem 10. *Which vector in* $\mathbb{R}^2$ *is* $\overrightarrow{a}$*? Why is this the only possible additive identity of* $\mathbb{R}^2$*?*

Axiom 5 (Inverse of vector addition)**.** *For a vector $\overrightarrow{u}$ in $\mathbb{R}^2$, there is a corresponding vector $\overrightarrow{v}$ such that $\overrightarrow{v} + \overrightarrow{u} = \overrightarrow{u} + \overrightarrow{v} = \overrightarrow{a}$ where $\overrightarrow{a}$ is the additive identity of $\mathbb{R}^2$. We call $\overrightarrow{v}$ the additive inverse of $\overrightarrow{u}$.*

Problem 11. *Consider a vector*

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

in $\mathbb{R}^2$. What is the additive inverse of $\overrightarrow{u}$? Can you write the additive inverse in terms of scalar multiplication? Why is there only one possible additive inverse of $\overrightarrow{u}$?

Axiom 6 (Closure of scalar multiplication)**.** *For a real number t and a vector $\overrightarrow{u}$ in $\mathbb{R}^2$, we know that $t \cdot \overrightarrow{u}$ is also in $\mathbb{R}^2$.*

Problem 12. *Prove that scalar multiplication on $\mathbb{R}^2$ is closed.*

Axiom 7 (Identity of scalar multiplication)**.** *There is a real number m such that $m \cdot \overrightarrow{u} = \overrightarrow{u}$ for any vector $\overrightarrow{u}$ in $\mathbb{R}^2$.*

Problem 13. *Which real number is m? Why is this the only possible choice for m?*

Axiom 8 (Associativity of scalar multiplication)**.** *For real numbers* t, s*, we have that* $(t \cdot s) \cdot \overrightarrow{u} = t \cdot (s \cdot \overrightarrow{u})$ *for any vector* $\overrightarrow{u}$ *in* $\mathbb{R}^2$*.*

Problem 14. *Prove that scalar multiplication on* $\mathbb{R}^2$ *is associative.*

Axiom 9 (Distributivity across scalar addition)**.** *For real numbers* t, s *and a vector* $\overrightarrow{u}$ *in* $\mathbb{R}^2$*, we know that* $(t + s) \cdot \overrightarrow{u} = (t \cdot \overrightarrow{u}) + (s \cdot \overrightarrow{u})$*.*

Problem 15. *Prove that scalar multiplication on* $\mathbb{R}^2$ *distributes across scalar addition in* $\mathbb{R}$*.*

Axiom 10 (Distributivity across vector addition)**.** *For a real number* t *and vectors* $\overrightarrow{u} \cdot \overrightarrow{v}$ *in* $\mathbb{R}^2$*, we know that* $t \cdot (\overrightarrow{u} + \overrightarrow{v}) = (t \cdot \overrightarrow{u}) + (t \cdot \overrightarrow{v})$*.*

Problem 16. *Prove that scalar multiplication on* $\mathbb{R}^2$ *distributes across vector addition in* $\mathbb{R}^2$*.*

Remark 2. *Note that we do not multiply two vectors in* $\mathbb{R}^2$ *together. Instead, we can only scale a vector, which is multiplying that vector by a real number. The reason comes from physics intuition, as forces in the real world do not multiply but rather add together. You may correctly argue "since when has the real world stopped mathematicians", and you'd be right. It is simply a question of mathematical modeling; ask your instructor or Google about "algebras"!*

General vector spaces

Wow, those proofs were super simple! They all only rely on the twelve algebraic axioms that the real numbers satisfy. In other words, they all hold true because $\mathbb{R}$ is a field. This begs the question, what is special about $\mathbb{R}$ and two dimensions? As you may guess, nothing! This allows us to generalize to abstract vector spaces while maintaining the same ten axioms we'd like.

Definition 7. *Let $d \geq 1$ be an integer and let $\mathbb{F}$ be a field with addition $+$ and multiplication $\cdot$. A d-dimensional $\mathbb{F}$-vector is a tuple of d-many numbers $u_1, \ldots, u_d$ in $\mathbb{F}$ written as*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix}.$$

We define the d-dimensional vector space over $\mathbb{F}$ to be the set of all d-dimensional $\mathbb{F}$-vectors, which we denote by $\mathbb{F}^d$. We define the binary operation of addition $+$ on $\mathbb{F}^d$ as

$$\begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_d + v_d \end{bmatrix}.$$

We define the binary operation of scalar multiplication $\cdot$ on $\mathbb{F}$ and $\mathbb{F}^d$ as

$$t \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} = \begin{bmatrix} tu_1 \\ \vdots \\ tu_d \end{bmatrix},$$

for t in $\mathbb{F}$. In this context, we typically call t a scalar. Note that we usually drop the symbol $\cdot$ and simply write

$$t \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} = \begin{bmatrix} tu_1 \\ \vdots \\ tu_d \end{bmatrix}$$

when the context is clear.

Remark 3. *Since $\mathbb{F}$ is a field, the proofs of Axiom 1 through Axiom 10 still hold true when replacing $\mathbb{R}^2$ with $\mathbb{F}^d$ everywhere. This is awesome!*

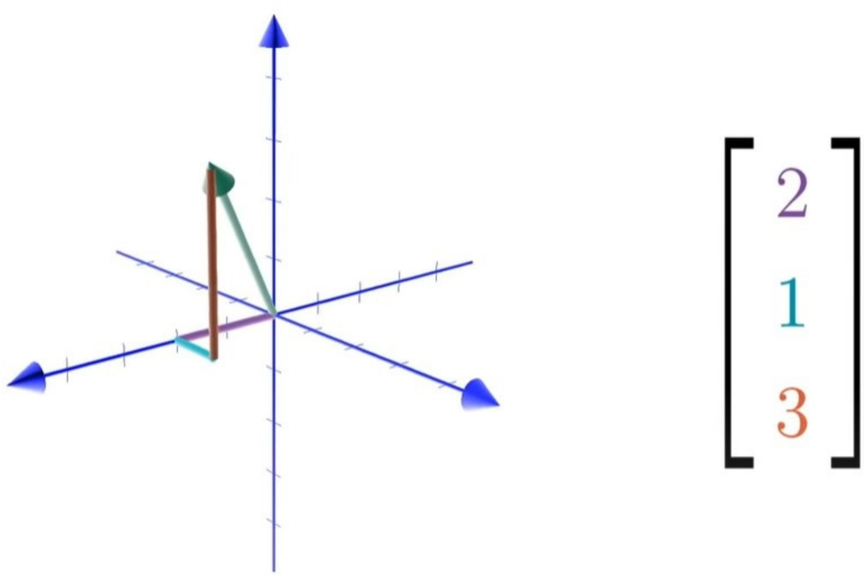
$\mathbb{R}^2$ is of course an example of Definition 7 for the case of $d = 2$ and $\mathbb{F} = \mathbb{R}$. Since we know that $\mathbb{R}^2$ models the Euclidean 2-dimensional plane, we may ask: what other kinds of geometric spaces can vector spaces model?

Example 1. *Consider the case of $d = 1$ and $\mathbb{F} = \mathbb{R}$ in Definition 7. $\mathbb{R}^1$ is the set of all 1-dimensional real vectors*

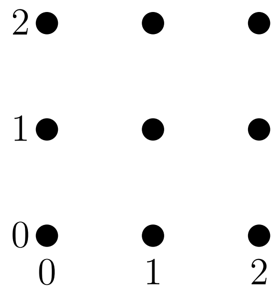
$$\begin{bmatrix} x \end{bmatrix}$$

for x in $\mathbb{R}$. One clearly sees that $\mathbb{R}^1$ is simply equal to $\mathbb{R}$ with the numbers just written in-between brackets. We know then that $\mathbb{R}^1 = \mathbb{R}$ models a 1-dimensional line, which is the standard number line that we use every day.

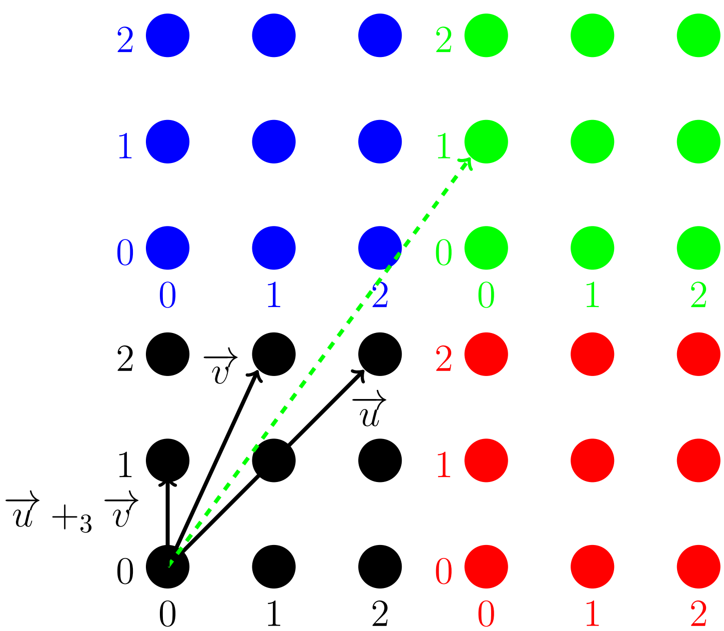
Example 2. *Consider the case of $d = 3$ and $\mathbb{F} = \mathbb{R}$ in Definition 7. $\mathbb{R}^3$ is the set of all 3-dimensional real vectors which models 3-dimensional Euclidean space, as seen in this screenshot from the YouTube video (with colors inverted to save printer ink).*



Example 3. *Consider the case of $d = 2$ and $\mathbb{F} = \mathbb{Z}_3$ in Definition 7. $(\mathbb{Z}_3)^2$ is the set of all 2-dimensional $\mathbb{Z}_3$-vectors which we can geometrically describe as the following lattice.*



When you add two vectors together, i.e move on this lattice, you wrap around like in an old video game. One way to understand this visually is to extend the lattice as we've done below. Wherever you land in a sub-lattice corresponds to positions in the original $(\mathbb{Z}_3)^2$ black sub-lattice.



For example, when you add $\overrightarrow{u}, \overrightarrow{v}$ as though they are vectors in $\mathbb{R}^2$, you obtain the dashed green vector. This corresponds to the vector $\overrightarrow{u} +_3 \overrightarrow{v}$ in the original $(\mathbb{Z}_3)^2$ black sub-lattice. Ask your instructors to draw some more examples!

Problem 17. *Can you express this wrapping in terms of addition modulo 3?*

These examples give a glimpse into the modeling power of vector spaces. Using $\mathbb{R}$ as the field, we can model the standard one, two, and three-dimensional spaces but we can also meaningfully work in higher dimensions even though we cannot imagine or see higher dimensions. Using modular fields such as $\mathbb{Z}_3, \mathbb{Z}_5$, or $\mathbb{Z}_7$, we can also work with d-dimensional spaces that wrap around themselves. Let's finish this handout with some computational problems to get familiar with vector spaces.

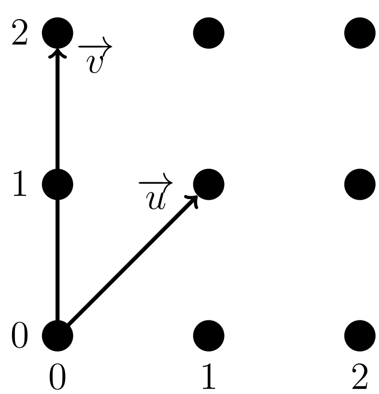
Problem 18. *Evaluate the following vector sums and scalings in* $\mathbb{R}^3$.

$$\sqrt{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \qquad \tfrac{1}{2}\begin{bmatrix} 7.1 \\ 1.1 \\ -1 \end{bmatrix} + \tfrac{3}{2}\begin{bmatrix} 7.2 \\ 1.2 \\ -0.9 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Problem 19. *Evaluate the following vector sums and scalings in* $\mathbb{R}^5$.

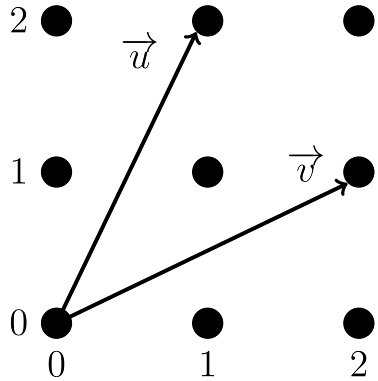
$$\sqrt{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \\ 2 \\ 1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ 2 \\ 1 \\ \sqrt{8} \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \\ \quad \end{bmatrix} \qquad \tfrac{1}{2}\begin{bmatrix} 7.1 \\ 1.1 \\ -1 \\ 4 \\ 8 \end{bmatrix} + \tfrac{3}{2}\begin{bmatrix} 7.2 \\ 1.2 \\ -0.9 \\ \frac{1}{3} \\ \frac{7}{3} \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

Problem 20. *Fill in the coordinates of the vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $(\mathbb{Z}_3)^2$ *depicted on the lattice below. Then compute the coordinates of* $2\overrightarrow{u} +_3 \overrightarrow{v}$ *in* $(\mathbb{Z}_3)^2$ *and draw the result on the lattice.*



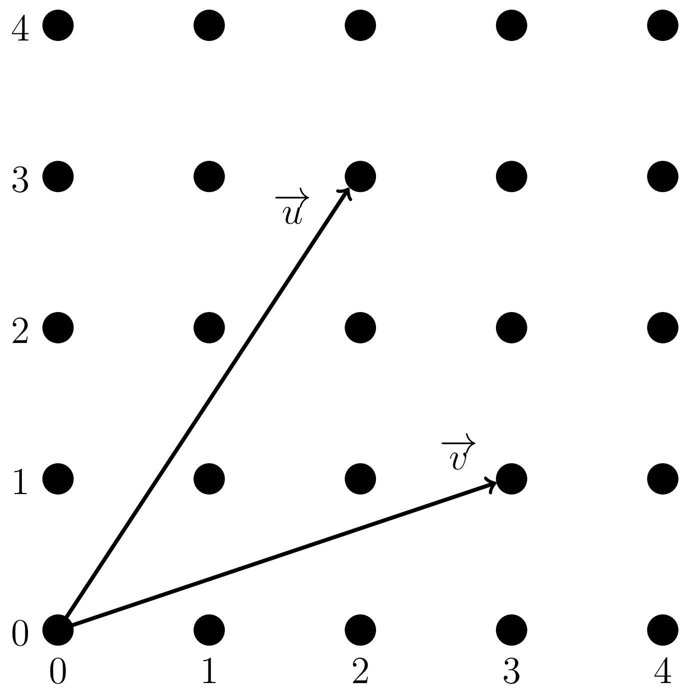
$$\overrightarrow{u} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \quad 2\overrightarrow{u} +_3 \overrightarrow{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Problem 21. *Fill in the coordinates of the vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $(\mathbb{Z}_3)^2$ *depicted on the lattice below. Then compute the coordinates of* $2\overrightarrow{u} +_3 2\overrightarrow{v}$ *in* $(\mathbb{Z}_3)^2$ *and draw the result on the lattice.*



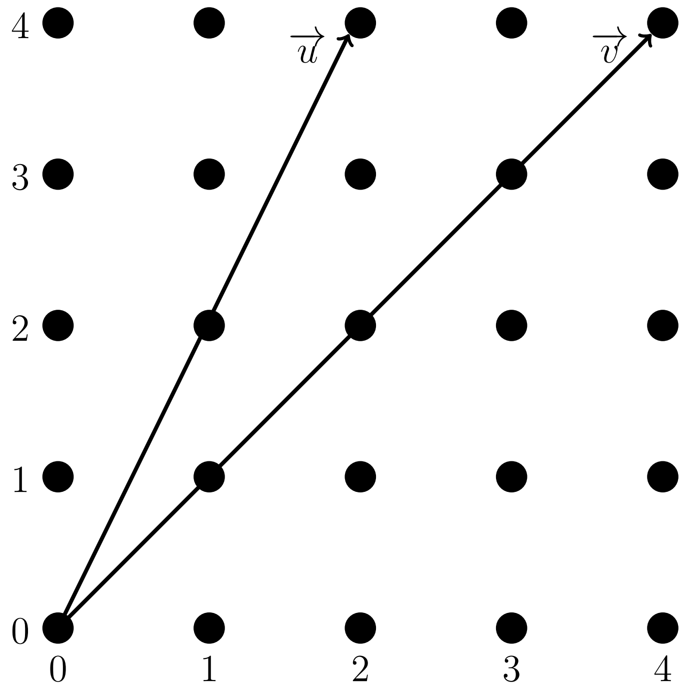
$$\overrightarrow{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix} 2\overrightarrow{u} +_3 2\overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Problem 22. *Fill in the coordinates of the vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $(\mathbb{Z}_5)^2$ *depicted on the lattice below. Then compute the coordinates of* $4\overrightarrow{u} +_5 2\overrightarrow{v}$ *in* $(\mathbb{Z}_5)^2$ *and draw the result on the lattice.*



$$\overrightarrow{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix} 4\overrightarrow{u} +_5 2\overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Problem 23. *Fill in the coordinates of the vectors* $\overrightarrow{u}, \overrightarrow{v}$ *in* $(\mathbb{Z}_5)^2$ *depicted on the lattice below. Then compute the coordinates of* $\overrightarrow{u} +_5 3\overrightarrow{v}$ *in* $(\mathbb{Z}_5)^2$ *and draw the result on the lattice.*



$$\overrightarrow{u} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \overrightarrow{u} +_5 3\overrightarrow{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Problem 24. *Evaluate the following vector sums and scalings in* $(\mathbb{Z}_7)^5$.

$$6\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} +_7 3\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \qquad 4\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} +_7 2\begin{bmatrix} 0 \\ 4 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$