ORMC AMC 10/12 Group
Sequences & Series

January 29, 2023

1 Arithmetic Sequences and Series

In general, a *sequence* is an ordered set of numbers, like $a_1, a_2, a_3, \ldots, a_n$.

An **arithmetic sequence** is one in which the difference between consecutive terms is constant. That is, the sequence $a_1, a_2, \cdots, a_n$ is arithmetic when $a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$. We often denote this common difference as d. Since every term is the same distance from the term before it, every term can be written as the first term, plus a multiple of the common difference. In general, the n^{th} term of an arithmetic sequence is:

$$\boxed{a_n = a_1 + (n-1)d}.$$

In general, every sequence has an associated *series*, which is the sum of the terms in the sequence. We often write a series using *summation notation*:

$$\sum_{n=1}^{m} a_n := a_1 + a_2 + a_3 + a_4 + \cdots + a_m.$$

The Σ indicates to substitute each of the integers from the lower summation bound to the upper bound, for n, and add all of those together. In the example above, the lower summation bound is $n = 1$, and the upper bound is m.

In general, if an arithmetic sequence has terms $a_1, \ldots, a_n$ and common difference d, then the sum of those terms is:

$$\boxed{n\frac{a_1 + a_n}{2}} = n\frac{a_1 + (a_1 + (n-1)d)}{2} = \boxed{n\frac{2a_1 + (n-1)d}{2}}.$$

Proof. We may write out the summation in terms of a_1 and d:

$$S = a_1 + (a_1 + d) + \cdots + (a_1 + (n-1)d).$$

We may similarly write it out in terms of a_n and d (and in reverse order):

$$S = a_n + (a_n - d) + \cdots + (a_n - (n-1)d).$$

If we pair up the terms, we get:

$$2S = (a_1 + a_n) + (a_1 + d + a_n - d) + \cdots + (a_1 + (n-1)d + a_n - (n-1)d) = n(a_1 + a_n)$$

$$\implies S = n\frac{a_1 + a_n}{2}.$$

□

The second boxed expression can be derived from writing a_n in terms of a_1 and d.

Note that it only makes sense for us to ask about a series for finite arithmetic sequences, because the terms an infinite arithmetic sequence grow infinitely, in proportion to d. So the sum of an infinite arithmetic sequence will always be $\pm\infty$.

1.1 Examples

1. **(2004 AMC 12B #8)** A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

2. **(2006 AMC 10A #19)** How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

3. **(1984 AIME #1)** Find the value of $a_2 + a_4 + a_6 + a_8 + \ldots + a_{98}$ if a_1, a_2, $a_3 \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$.

1.2 Exercises

1. **(2005 AIME I #2)** For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, 10, \ldots$. For how many values of k does S_k contain the term 2005?

2. **(1989 AIME #7)** If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

3. **(HMMT 2002 #36)** Find the set consisting of all real values of x such that the three numbers $2^x, 2^{x^2}, 2^{x^3}$ form a non-constant arithmetic progression (in that order).

2 Geometric Sequences and Series

A geometric sequence is one in which consecutive terms share a common ratio. That is, the sequence $a_1, a_2, \ldots, a_n$ is geometric when $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}}$. The common ratio is often denoted as r. Similar to arithmetic sequences, every term is the same multiple of the term before it, so we can write each term as the first term multiplied by some power of the common ratio. In general, the n^{th} term of a geometric sequence is

$$\boxed{a_n = a_1 r^{n-1}}.$$

The terms of a geometric sequence $a_1, a_2, \ldots, a_n$ with common ratio r always add up to

$$\boxed{a_1 \frac{r^n - 1}{r - 1}}.$$

Proof. The most important part of this proof is to recall the difference-of-powers factorization:

$$(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Then, we write the summation with all elements in terms of a_1 and r:

$$S = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} = a_1(1 + r + r^2 + \cdots + r^{n-1})$$

By the factorization above, this is equal to:

$$a_1 \frac{r^n - 1}{r - 1}.$$

□

When the common ratio r has absolute value less than 1, it might make sense to consider an infinite geometric series. As the terms get smaller, it is likely that the summation changes less and less, and it converges to some single value, as opposed to blowing up to $\pm\infty$. If $|r| < 1$, consider what happens if we make n larger and larger. Since $|r| < 1$, $|rx| < |x|$ for any x, so as n gets larger, $|r^n|$ must continue to get smaller, and it can never stay larger than any number other than 0. So, the numerator of the summation formula above will get closer and closer to $0 - 1$, while the rest of the expression remains unchanged. So, an infinite geometric series with first term a_1 and common ratio r has a sum

$$a_1 \frac{-1}{r - 1} = \boxed{a_1 \frac{1}{1 - r}}$$

as we add up the infinitely many terms.

2.1 Examples

1. **(2012 AIME II #2)** Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

2. **(2009 AIME I #1)** Call a 3-digit number geometric if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2.2 Exercises

1. **(2016 AMC 10B #16)** The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

2. **(2005 AIME II #3)** An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

3. **(2019 HMMT #5)** Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

4. **(2007 AIME II #12)** The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308 \text{ and } 56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57,$$

find $\log_3(x_{14})$.

3 Recursive Sequences

As we have mentioned before when working with recursive functions, something is recursive when it is defined in terms of previous iterations of itself. For example, you likely know the Fibonacci sequence, which has a famous recursive definition:

$$f_{n+2} = f_{n+1} + f_n, \quad f_0 = 0, f_1 = 1.$$

In this definition, we have two important parts. First, we have the recursion rule, $f_{n+2} = f_{n+1} + f_n$, which tells us the value of a term based on the previous two terms. Second, we have the base case, which tells us what values we should start with to derive the rest of the sequence with the recursion rule.

Note also that the arithmetic and geometric sequences can be written recursively:

$$\textbf{Arithmetic: } a_{n+1} = a_n + d, \quad a_1 = a_1$$
$$\textbf{Geometric: } a_{n+1} = r \cdot a_n, \quad a_1 = a_1$$

We have already discussed strategies to solve recurrence relations when they are given to you, such as: testing some initial terms, finding a pattern, seeing if/when the sequence repeats, simplifying the recurrence rule, etc.

However, some of the more interesting and difficult problems will require you to define the recurrence relations on your own. This can be useful when a problem asks you to count something that seems to depend heavily on smaller versions of itself. A good strategy for problems like this is to treat it as an element of a sequence, define a recurrence relation, and then solve for the relevant element of the sequence. A good way to approach constructing a recurrence relation is to single-out elements, generally the "last" one(s) Then, consider what happens to the rest of the elements when these "last" elements have a certain state.

Example. Let's say you have lots of pictures of the beach and at the park, and you have 7 picture frames up on the wall. You want to fill all the picture frames, but you don't want to have two pictures of the park next to each other. How many ways can you do this?

Solution. First, let's treat this as the 7^{th} element of a sequence, where the n^{th} element (a_n) of the sequence is the number of ways for you to fill n picture frames according to your rules.

The way you choose to fill the first $n-1$ frames greatly impacts how you can fill the n^{th} frame, and vice versa. So, this is a good candidate for recursion.

If you put a park picture in the n^{th} frame, then you must put a beach picture in the $n-1^{\text{th}}$ frame. Then, you can fill the remaining $n-2$ frames in any way you like, as long as it follows the rules. So, it is as if you just had $n-2$ frames to fill, which can be done in a_{n-2} ways.

On the other hand, if you put a beach picture in the n^{th} frame, this has no effect on the other frames. So, you can fill them any way you like, and it is just as if you had only $n-1$ frames to fill, which can be done in a_{n-1} ways.

Surprisingly, this sets up our recurrence relation! We have covered all the cases for the n^{th} frame, which means if we add up all the cases, we get a_n. The first case is a_{n-2}, and the second case is a_{n-1}, so the recursion rule is $a_n = a_{n-1} + a_{n-2}$.

As for the base case, we can fill 1 lone frame with either a park or beach picture, so $a_1 = 2$. And, we can fill 2 frames with any 2 pictures as long as they are not both park pictures, so $a_2 = 3$.

If we follow the recursion, we get that $a_7 = \boxed{34}$

In general, there are some good strategies to follow when defining a recurrence relation:

- As mentioned before, single out an element, usually the "last" one, and define your relation based on that one
- Keep it simple! If there are too many terms in your recurrence, it will be difficult to solve. (and since you're doing the AMC/AIME, it shouldn't be too complicated)
- If it starts getting complicated, consider breaking it up into separate sequences, or look for a different element that might give you a simpler recurrence. (you will see examples of these below)

3.1 Examples

1. **(2019 AMC 10B #25)** How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

2. **(2019 AMC 10A #15)** A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

3.2 Exercises

1. **(2007 AMC 12A #25)** Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

2. **(2008 AIME I #11)** Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

3. **(1999 HMMT A9)** How many ways are there to cover a 3×8 rectangle with 12 identical dominoes?

4. **(2006 AIME I #11)** A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

   Any cube may be the bottom cube in the tower. The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$. Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?