ORMC Intermediate 2A **January 29, 2023**

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Permutations and the 15 puzzle

The ultimate goal of the mini-course we begin now is to learn solving the *15 puzzle* (when a solution exists).

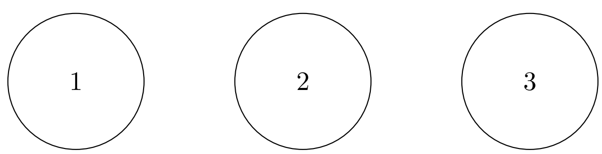
The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares in the proper order, left to right, starting with the top row as on the picture below.

15 puzzle

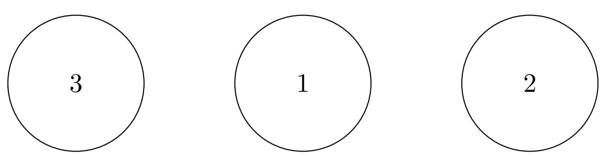
Problem 1 *Warm up: play around with the 15 Puzzle. Think about how you might be able to represent a given configuration of the puzzle and if that configuration is solvable.*

The mathematical foundation of the solution is the theory of *permutations*. The theory not only helps to unravel the puzzle, but also comes quite handy in a wide variety of applications, from card tricks to quantum mechanics. It will take us a few classes to learn the basics. In the meantime, you are encouraged to get the puzzle, either in the solid form or as a smartphone/tablet app, and to start playing!

Consider a set of marbles numbered 1 through n. Originally the marbles are lined up in the order given by their numbers. The following picture shows an example with $n = 3$.



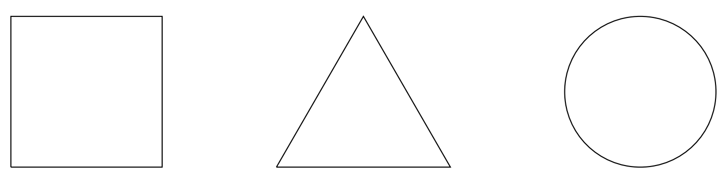
Then the marbles are reshuffled in a different order.



A *permutation* is the operation of reshuffling the marbles (or elements of any set). The one shown in the example is written down as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

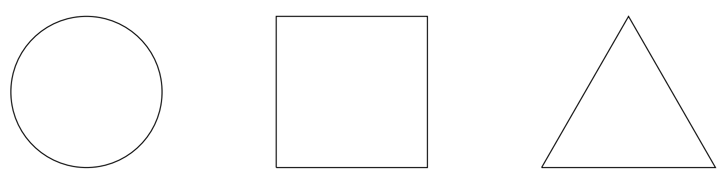
This notation can be read in the following way. Starting in the top row, we look at the first number, 1. Looking below it, we see the marble numbered 1 ends in the position where the marble numbered 2 starts. Then we move over one column to see the number 2 on top and the number 3 below it, so the marble in position 2 ends in the position where the marble numbered 3 starts. Finally, we see in the last column that the marble numbered 3 ends in the position where the marble numbered 1 starts. Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let us consider the following geometric figures rather than the marbles numbered 1, 2, and 3.



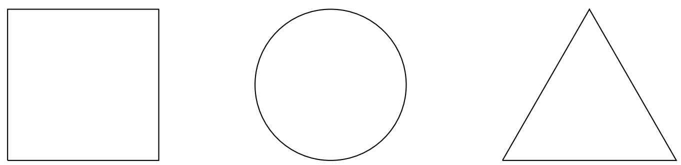
Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

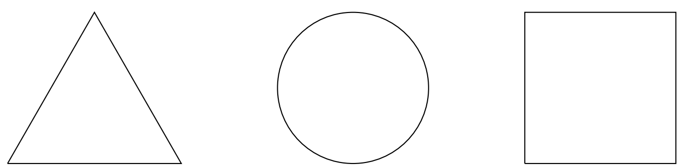
will reshuffle the figures into the following order.



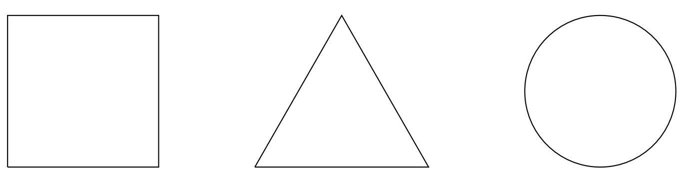
Problem 2 *For the original order of figures given on page 3, write down the permutations that correspond to the following pictures.*



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that the last permutation does not reshuffle anything at all. Permutations of this kind, typically denoted as e, are called *trivial*. A trivial permutation is still a permutation, and an important one!

Problem 3 *Write down the trivial permutation for $n = 5$.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Problem 4 *For the original order of figures given on page 3 draw the figures in the orders prescribed by the permutations below. Use the space to the right of a permutation to draw the corresponding picture.*

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Recall that $n! = 1 \times 2 \times \ldots \times (n-1) \times n$. For example, $3! = 1 \times 2 \times 3 = 6$.

Problem 5 *Compute* 5!

$$5! =$$

Problem 6 *How many permutation of four elements are there?*

Problem 7 *How many permutation of $n+1$ elements are there?*

Problem 8 *Write down a permutation of four elements.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 9 *Write down a permutation of four elements that keeps the third element in place.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Find the number of permutations of four elements that keep the third element in place.

$$\textit{The number} =$$

It is possible to combine, or *multiply*, permutations. For example, let us apply the permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

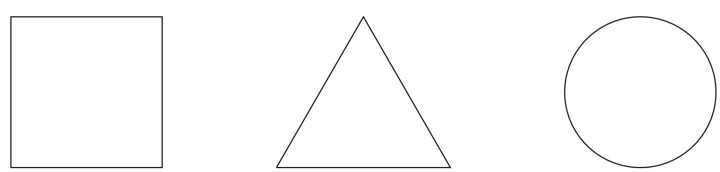
to the marbles already reshuffled by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutation δ switches the first and second elements, so

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

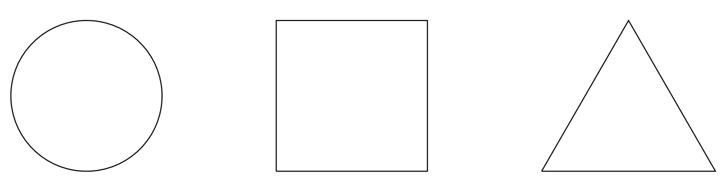
Let us take another look at the above computation using the figures from page 3. Originally, the set of the figures is ordered as follows.



The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

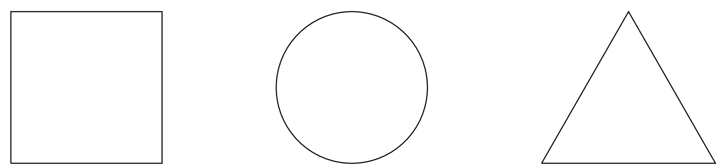
produces the picture below.



The permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

applied to the latter configuration gives us the following.



Comparing the last picture to the original gives us the answer.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note that in the product $\delta \circ \sigma$ of permutations, it is the one on the right, σ, that acts first on the set it permutes!

Problem 10 *Find the permutation $\sigma \circ \delta$. If needed, use the pictorial representation as above.*

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

We say two permutations σ and δ **commute** *if $\sigma \circ \delta = \delta \circ \sigma$. Do σ and δ given above commute?*

Note that although some particular permutations may commute, multiplication of permutations in general is not a commutative operation!

Problem 11 *Find two non-trivial permutations of four elements that do commute.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 12 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

If you need to use a pictorial representation as a tool, take the one on page 3 and add a diamond $\diamondsuit$ as the fourth figure.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 13 *Find the product $\sigma \circ \delta$ of the permutations*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \textit{and} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

from Problem 12. If needed, use a pictorial representation.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

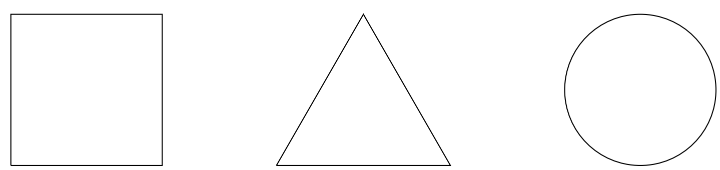
Do the permutations δ and σ commute?

A permutation δ is called *opposite* to a permutation σ if

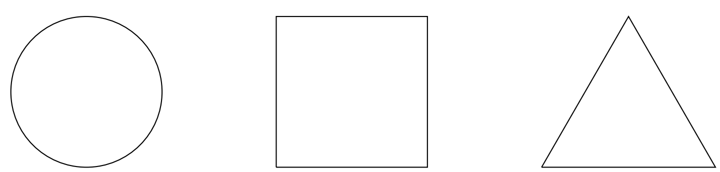
$\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse*.

Example 1 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The permutation σ *reshuffles the figures*



in the following order.



Hence, $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

Problem 14 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. *Check your work by multiplying* $\sigma \circ \sigma^{-1}$ *and* $\sigma^{-1} \circ \sigma$ *to ensure they both equal* e.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that since the permutation σ^{-1} undoes what the permutation σ does, σ works the same way for σ^{-1}. Hence, not only $\sigma^{-1} \circ \sigma = e$, but $\sigma \circ \sigma^{-1} = e$ as well. Thus, σ and σ^{-1} always commute.

$$\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e \tag{1}$$

Problem 15 *Find* σ^{-1} *for*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Check your work by multiplying $\sigma \circ \sigma^{-1}$ *and* $\sigma^{-1} \circ \sigma$ *to ensure they both equal* e*.*

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problems 14 and 15 exhibit two different non-trivial permutations σ that are self-inverse, $\sigma^{-1} = \sigma$.

Problem 16 *Find a non-trivial permutation* σ *different from the ones in Problems 14 and 15 such that* $\sigma^{-1} = \sigma$*.*

The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a prominent American chess player at the time,[1] has offered $1,000 (about $25,000 of modern day money) for solving the puzzle in the form shown on the picture below.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Proving that this particular configuration has no solution will be the primary goal of our mini-course.

Problem 17 *Write down the permutation corresponding to the Loyd's puzzle, when compared to the configuration given on Page 1.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ & & & & & & & & & & & & & & \end{pmatrix}$$

[1]Ranked 15th in the world.

Sam Loyd, 1841 – 1911

Problem 18 *Let us call σ the permutation from Problem 17. Find σ^{-1}.*

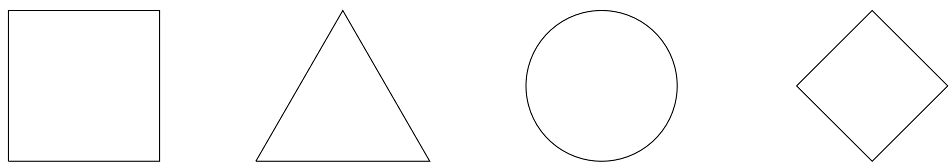
$$\sigma^{-1} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ & & & & & & & & & & & & & & \end{pmatrix}$$

Problem 19 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

If you cannot do it right away, please use the following pictorial representation for the original arrangement.



$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Note that the first line of the notation we have used for writing down permutations so far is redundant. Indeed

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

means that we shuffle the second element to the first position, the first element to the second position, the fourth element to the third position, and the third element to the fourth one. Without any loss of clarity, we can write this down as

$$\sigma = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}.$$

Problem 20 *Apply the permutation* $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ *to the sequence of geometric figures on page 3 and draw the result in the space below.*

Problem 21 *Find the product* $\delta \circ \sigma$ *of the following two permutations.*

$$\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} \qquad \delta = \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix}$$

$$\delta \circ \sigma = \begin{pmatrix} \qquad\qquad\qquad\qquad \end{pmatrix}$$

Let us set $\sigma^0 = e$ for any permutation σ. The permutation σ^2 is defines as $\sigma \circ \sigma$, σ^3 as $\sigma \circ \sigma^2$, and so on. Similarly, $\sigma^{-2} = \sigma^{-1} \circ \sigma^{-1}$, $\sigma^{-3} = \sigma^{-1} \circ \sigma^{-2}$, and so forth.

Problem 22 *Find the following powers of the permutation* $\sigma = \left(3 \;\; 1 \;\; 4 \;\; 2 \right)$ *from Problem 21.*

$$\sigma^2 = \left(\qquad\qquad\qquad\qquad \right)$$

$$\sigma^3 = \left(\qquad\qquad\qquad\qquad \right)$$

$$\sigma^4 = \left(\qquad\qquad\qquad\qquad \right)$$

$$\sigma^{-1} = \left(\qquad\qquad\qquad\qquad \right)$$

$$\sigma^{-2} = \left(\qquad\qquad\qquad\qquad \right)$$

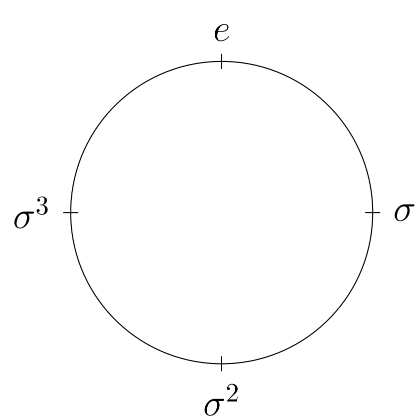
$$\sigma^{-3} = \left(\qquad\qquad\qquad\qquad \right)$$

$$\sigma^{-4} = \left(\qquad\qquad\qquad\qquad \right)$$

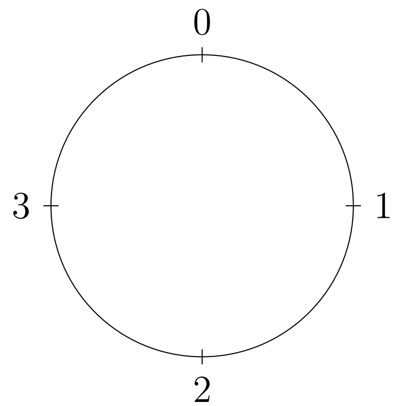
Solving Problem 22, you may have noticed the following. The formula $\sigma^4 = e$ means that

- $\sigma \circ \sigma^3 = e$, hence $\sigma^{-1} = \sigma^3$ and $\sigma^{-3} = \sigma$;
- $\sigma^2 \circ \sigma^2 = e$, hence $\sigma^{-2} = \sigma^2$. Furthermore,
- $\sigma^5 = \sigma^4 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^6 = \sigma^4 \circ \sigma^2 = e \circ \sigma^2 = \sigma^2$;
- $\sigma^7 = \sigma^4 \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$;
- $\sigma^8 = \sigma^4 \circ \sigma^4 = e \circ e = e$;
- $\sigma^9 = \sigma^8 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^{-5} = \sigma^{-4} \circ \sigma^{-1} = e^{-1} \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$; and so forth.

It turns out that all the powers of the permutation σ reside naturally on the following circle.



Consider the integers on a circle divided into four equal parts.



On the circle, 0 coincides with 4. We write this fact down as

$$4 \equiv 0 \ (mod\ 4) \tag{2}$$

and read it as *4 is congruent to 0 modulo 4*. The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The *mod* 4 symbol tells us that the circle is divided into 4 equal parts, so 4 coincides with 0, 5 with 1, 6 with 2, and so on. Or in the new notations, $4 \equiv 0 \ (mod\ 4)$, $5 \equiv 1 \ (mod\ 4)$, $6 \equiv 2 \ (mod\ 4)$, $7 \equiv 3 \ (mod\ 4)$, and so forth.

Problem 23

$$-21 \equiv \qquad\qquad (mod\ 4)$$

$$6 + 5 \equiv \qquad\qquad (mod\ 4)$$

As we can see, powers of the permutation σ from Problems 21 and 22 produce nothing more than a multiplicative realization of the *mod* 4 arithmetic. In other words, the *mod* 4 integers on the second circle serve as powers of the permutation σ on the first circle.

Example 2 *Find the 125 power of the permutation* σ *from Problems 21 and 22.*

$$\sigma^{125} = \sigma^{1 \ (mod \ 4)} = \sigma$$

Problem 24 *Find the -333 power of the permutation* σ *from Problems 21 and 22.*

$$\sigma^{-333} = (\qquad\qquad\qquad\qquad)$$

The smallest positive power n of a permutation δ such that $\delta^n = e$ is called the *order of the permutation.*

Problem 25 *What is the order of the permutation* σ *we have considered in Problems 21, 22, and 24?*

Problem 26 *Find the following powers of the permutation* $\mu = \begin{pmatrix} 3 & 2 & 4 & 1 \end{pmatrix}$.

$$\mu^2 = (\qquad\qquad\qquad)$$

$$\mu^3 = (\qquad\qquad\qquad)$$

$$\mu^4 = (\qquad\qquad\qquad)$$

$$\mu^{-1} = (\qquad\qquad\qquad)$$

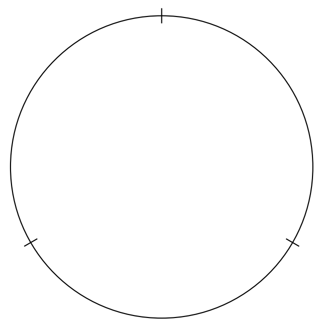
$$\mu^{-2} = (\qquad\qquad\qquad)$$

$$\mu^{-3} = (\qquad\qquad\qquad)$$

What is the order of the permutation μ*?*

The problem continues on the next page.

Mark μ^{123}, μ^{124}, *and* μ^{125} *on the circle below.*



What mod n *arithmetic is realized by the powers of* μ*?*