Hyperbolic Geometry II - Angles in the Hyperbolic Plane

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1 Hyperbolic Triangles

This week, we continue the example of the *hyperbolic plane* that we started with last week. Last week, we showed that in the hyperbolic plane, given any line and a point not on that line, there are infinitely many parallel lines through that point. We now continue developing this new geometry by studying shapes - the most basic of which is the *triangle* (a shape with three hyperbolic line segments for sides). (Technically, the most basic shape would have two sides, but that is impossible in the hyperbolic plane - think about why.) As a reminder:

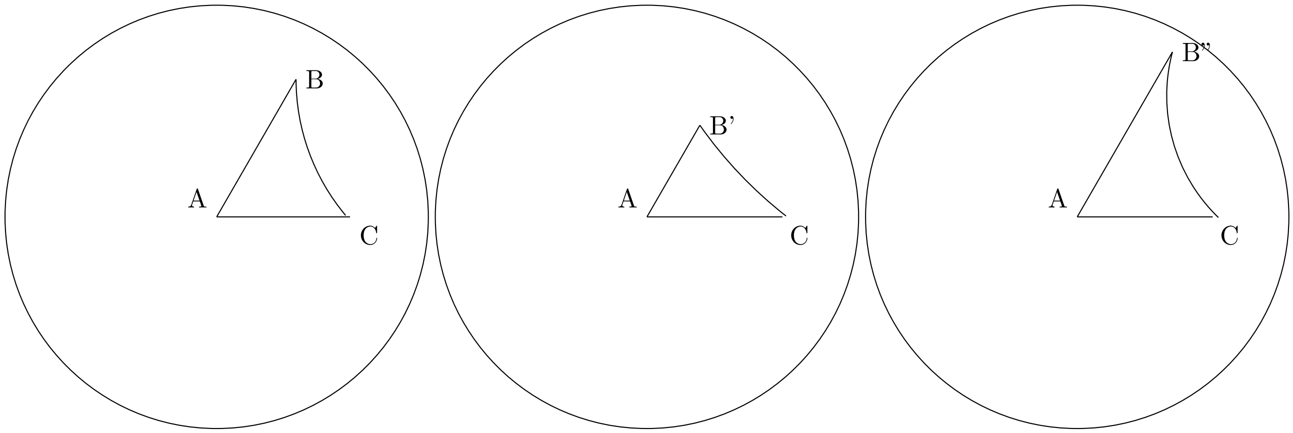
Definition 1 *In Poincaré's model of the hyperbolic plane, also known as the* ***Poincaré disk****:*

- *The points are all points on the plane that are less than* 1 *away from the origin in distance.*
- *The points on the unit circle* S^1 *(exactly* 1 *away from the origin in distance) are called the* ***ideal points****.*
- *The (hyperbolic) lines are given by circles orthogonal to* S^1*, as well as diameters of* S^1*. The angles between shapes in general is the angle between their tangent lines at the intersection point.*

Problem 1 *Show that the angles of a (non-degenerate) triangle add up to less than* 180 *degrees. (Hint: Last week, we learned that circle inversions preserve circles and angles. Use an inversion to move some sides of the triangle to diameters, as it is easier to find the angle between straight lines.)*

Solution: By inverting about the right circle, assume one vertex is at the origin. Then two edges are straight (they are along diameters) so the hyperbolic triangle is immediately comparable to the Euclidean triangle with those vertices. Since the hyperbolic triangle must be concave, its angles must be strictly smaller.

As it turns out, the angles of a hyperbolic triangle depend on the side lengths of the triangle. Those familiar with Euclidean geometry will recall a similar relationship in the plane. Before we quantify this relationship, let's look at it qualitatively. In the following diagrams, points A and C are the same points, while points B, B', and B'' lie on the same ray from point A. Each diagram shows a triangle, so AC, AB, AB', and AB'' are diameters.



Problem 2 *Like in the Euclidean case, we define "Angle B" to be the angle at vertex B (so in this case, angle ABC). Which of the angles B, B', and B'' is largest? Smallest? Is there a pattern?*

Solution: Angle B'' is smallest and angle B' is the largest. The longer AB is (at least in the Euclidean sense), the smaller angle B is.

Recall the hyperbolic distance function defined previously:

Definition 2 *Let A and B be points in the hyperbolic plane. Let P and Q be the ideal endpoints of the hyperbolic line connecting A and B such that P is closer to B and Q is closer to A. The* ***hyperbolic distance*** *from A to B is given by*

$$d(A, B) := \log([A, B; P, Q])$$

where the cross ratio is given by

$$[A, B; P, Q] = \frac{AP \cdot BQ}{AQ \cdot BP}$$

using the Euclidean distances in that formula.

Problem 3 *Fix A inside the disk, and take a point B approaching the boundary. Show that $d(A, B)$ gets larger and larger as you do this (we say that $d(A, B) \to \infty$).*

Solution: As B approaches the boundary, the distance BP becomes smaller and approaches zero.

This suggests a natural extension of the distance function we've defined.

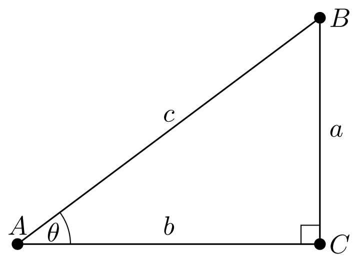
Definition 3 *If A or B is an ideal point, we say that $d(A, B) = \infty$.*

Problem 4 *Let ABC be a hyperbolic triangle with $d(A, B) = d(B, C) = \infty$. What is angle B?*

Solution: Angle B is zero, because since B is an ideal point, AB and BC must both be orthogonal to S^1 at point B.

2 The Euclidean Functions sin and cos

In the Euclidean case, there is likewise a relationship between the sides of a triangle and its angles. To quantify this, we use *trigonometric functions*, which are first defined for acute angles.



Definition 4 *The functions **sine** and **cosine** of the angle θ are defined by*

$$\sin(\theta) = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{a}{c} \ \textit{and} \ \cos(\theta) = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{b}{c}$$

Problem 5 *Calculate:*

- $\cos(30°)$

  Solution: $\sqrt{3}/2$.

- $\sin(45°)$

  Solution: $\sqrt{2}/2$.

- $\cos(60°)$

  Solution: $1/2$.

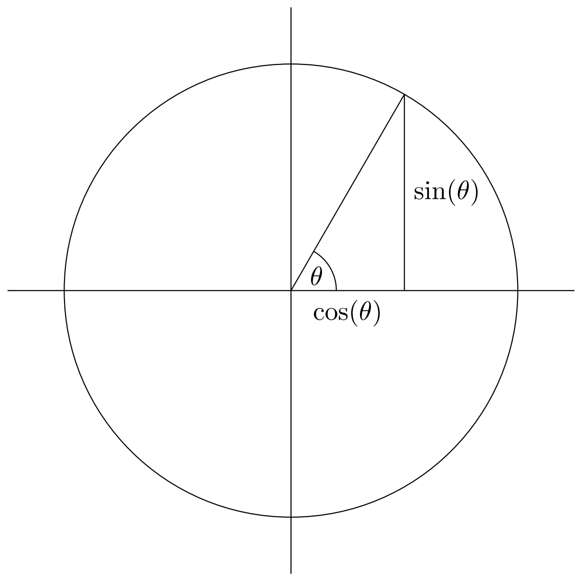
Problem 6 *Let θ be an acute angle. Show that*

$$\cos^2(\theta) + sin^2(\theta) = 1$$

Solution: Draw any right triangle (like the one above) with angle θ. Then by the Pythagorean Theorem,

$$\cos^2(\theta) + sin^2(\theta) = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{c^2}{c^2} = 1$$

Problem 6 shows that the functions sin and cos lie on the unit circle, like so:



This leads us to define the sine and cosine of any angle:

Definition 5 *The **cosine** of any angle is the x-coordinate of the intersection of the ray from the origin at that angle with the unit circle. The **sine** is the y-coordinate of that intersection.*

The third important trig function, which we will not study this week, is the *tangent*, which is just sine divided by cosine. The cosine function in particular allows us to find side lengths from angles, or vice versa.

Problem 7 *Calculate:*

- $\cos(135°)$

  Solution: $-\sqrt{2}/2$.

- $\sin(180°)$

  Solution: 0.

- $\cos(240°)$

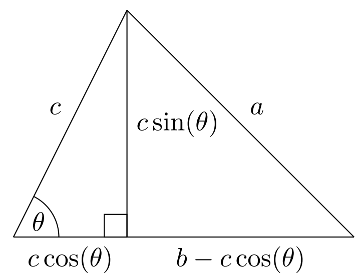
  Solution: $-1/2$.

Theorem 1 *(Euclidean Law of Cosines) Let A, B, C be the angles at the respective vertices and a, b, c be the (Euclidean) side lengths of the side lengths opposite to the respective vertices. Then*

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

Problem 8 *(Bonus) Prove Theorem 1.*

Solution: This is the case of an acute angle, but the obtuse case follows similarly. Drop the perpendicular from B to side AC, and find the lengths as follows:



Using the Pythagorean Theorem on the rightmost triangle shows that

$$a^2 = (c\sin(\theta))^2 + (b - c\cos(\theta))^2 = b^2 - 2bc\cos(\theta) + c^2\cos^2(\theta) + c^2\sin^2(\theta) = b^2 + c^2 - 2bc\cos(\theta)$$

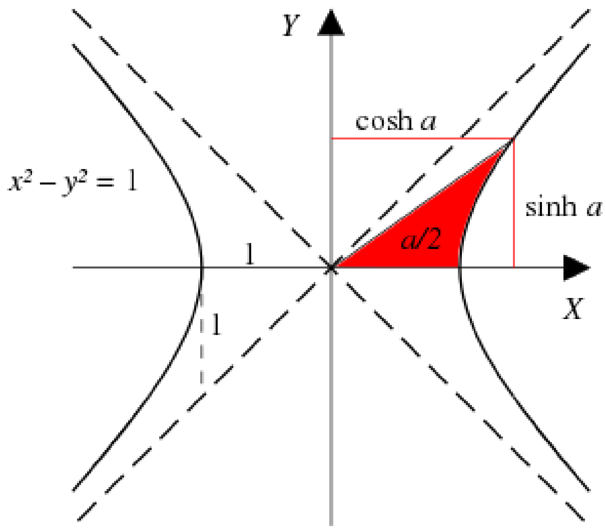
and rearranging gives the right form.

Problem 9 *Consider increasing a, b, c. We know by the triangle inequality that we better increase two at a time, so consider all the ways to do that. What happens to angle A as you keep increasing a and b? What about a and c? And finally, b and c? (Hint: Consider all the possible angles of a triangle - is cosine increasing on this domain, decreasing, or neither?)*

Solution: Since cosine decreases from 0 to 180 degrees (no Euclidean triangle can have a larger angle), as the right-hand side gets larger the left-hand side gets smaller, and vice versa. Increasing a and b makes the right-hand side go to zero, so angle A becomes a right angle, and same with increasing a and c. Increasing b and c makes the right-hand side closer to 1 so angle A approaches zero.

3 The Functions sinh and cosh

Just like the functions sine and cosine are defined by rays intersecting the circle $x^2 + y^2 = 1$, we can define the *hyperbolic sine and cosine* (abbreviated sinh and cosh respectively) by rays intersecting the hyperbola $x^2 - y^2 = 1$. This is illustrated below, but is unwieldy to calculate this way, so we'll use a formula (which is equivalent to finding the rays) instead.



Definition 6 *Given any real number a, the functions* $\sinh(a)$ *and* $\cosh(a)$ *are given by*

$$\cosh(a) = \frac{e^a + e^{-a}}{2} \text{ and } \sinh(a) = \frac{e^a - e^{-a}}{2}$$

Additionally, just like with the Euclidean sine and cosine, the ***hyperbolic tangent*** *is defined by* $\tanh(a) = \sinh(a)/\cosh(a)$.

Problem 10 *When is* $\cosh(a) > 0$*? When is* $\sinh(a) > 0$*?*

Solution: $\cosh(a)$ is always positive (in fact, it's always at least 1, but that's much harder to prove) because $e^a, e^{-a} > 0$. $\sinh(a) > 0$ if and only if $e^a > e^{-a}$, which is true if and only if $a > 0$ (for example, by multiplying both sides).

Problem 11 *Find the range of* tanh.

Solution: Because e^{-a} is always positive, $|\sinh(a)| \leq \cosh(a)$ so $|\tanh(a)| < 1$. To see that the range is $(-1, 1)$, take a very large and very small.

Problem 12 *Show that* tanh *is a bijection, and find* $\tanh^{-1}$. *(Hint: Recall that any function with an inverse that's defined for all values in the range is a bijection. To find the inverse, set* $\tanh(y) = x$ *and solve for* y*.)*

Solution: tanh is a bijection because it has an inverse - namely,

$$\tanh^{-1}(a) = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$$

which is defined on $(-1, 1)$.

Problem 13 *Show that in the hyperbolic plane, given any point* A *which is Euclidean distance* r *from the origin* O*, the hyperbolic distance satisfies*

$$d(O, A) = 2\tanh^{-1}(r)$$

Solution: Using the previous problem and the fact that $OP = OQ = 1$,

$$d(O, A) = \log[O, A; P, Q] = \log\left(\frac{(1+r)1}{1(1-r)}\right) = 2\tanh^{-1}(r)$$

4 Angles of Hyperbolic Triangles

Just like in the Euclidean case, we use the convention that angles A, B, C of hyperbolic triangle ABC are at the corresponding vertices, and a, b, c are the hyperbolic side lengths opposite vertices A, B, C, respectively. We'll make use of the following theorem:

Theorem 2 *(Hyperbolic Law of Cosines)*

$$\cos(A) = \frac{\cosh(b)\cosh(c) - \cosh(a)}{\sinh(b)\sinh(c)}$$

Problem 14 *Prove Theorem* 2.

Solution: By a circle inversion, assume that vertex A is at the origin. Rotating if necessary, we assume that point B is on the x-axis, so that its coordinates are $(\tanh(c/2), 0)$ by Problem 13. Since point C is on a ray that is angle A away from AB, by definition its coordinates are $\tanh(b/2)\cos(A), \tanh(b/2)\sin(A)$. To calculate a in terms of these values, invert C to the origin and use Problem 13 again, which gives that

$$a = 2\tanh^{-1}\left(\sqrt{\frac{(\tanh(c/2) - \tanh(b/2)\cos(A))^2 + (\tanh(b/2)\sin(A))^2}{(\tanh(c/2)\tanh(b/2)\cos(A) - 1)^2 + (\tanh(c/2)\tanh(b/2)\sin(A))^2}}\right)$$

Now we manipulate

$$\cosh(a) = \cosh(2(a/2)) = \cosh^2(a/2) + \sinh^2(a/2) = \frac{1 + \tanh^2(a/2)}{1 - \tanh^2(a/2)}$$

and following the algebra will lead to the correct expression.

Problem 15 *Like in problem* 9*, consider increasing b and c. (This corresponds to moving vertex A towards the boundary.) What happens to angle A? Compare to your answer to Problem* 4.

Solution: Increasing both b and c makes the right-hand side tend to 1, and therefore angle A should tend to zero. This matches our answer to Problem 4, as once A becomes an ideal point ($b = c = \infty$), angle A will be zero.

Problem 16 *(Challenge) What happens when you increase a and b as in Problem 9?*

Problem 17 *In Problem 1, we showed that the sum of angles of any hyperbolic triangle are less than 180 degrees. How does the sum of angles depend on the side lengths?*

Solution: Various answers. Students should notice that increasing all the side lengths decreases the sum of the angles.