Hyperbolic Geometry I

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1 Non-Euclidean Geometries

Euclid's *Elements*, often considered the first mathematical treatise, begins with axioms that are used to prove theorems that are familiar to us about shapes in the plane. One of Euclid's axioms was as follows. Although Euclid himself stated it differently, it is equivalent to the following formulation:

Definition 1 *The* ***parallel postulate*** *states that given any line L and any point P not on L, there exists a unique line through P parallel to L.*

Euclidean geometry describes the *plane*, which can be expressed as the coordinate plane $\mathbb{R}^2$ - that is, ordered pairs of real numbers.

Problem 1 *Show that the coordinate plane $\mathbb{R}^2$ satisfies the parallel postulate. (Hint: Write lines in point-slope form. When are they parallel?)*

When studying *non-Euclidean geometry*, while we could negate any of Euclid's axioms, the most interesting examples occur when we negate the parallel postulate. The negation of "there exists a unique" is "there does not exist or there exists more than one", so we have two options for replacement axioms:

Definition 2 *The two alternative parallel postulates are:*

- *No two lines are parallel.*
- *Given any line L and point P not lying on L, there are at least two lines through P parallel to L.*

This week we will study one particular example satisfying the second of these.

2 Circle Inversions

We begin with a useful theorem.

Theorem 1 *(Power of a Point): Let ω be a circle, N a point not on ω, and L any line through N which intersects ω at two points M and M'. (Note: If L is tangent to ω, we consider the tangency point to be both M and M') Then the value $NM \cdot NM'$ is the same for any choice of line L.*

Problem 2 *(Bonus) Prove Theorem 1.*

Definition 3 *Let ω be a circle with center O and radius r. The inversion of any point $M \neq O$ in the plane (denoted $i_\omega(M)$) is the point N lying on ray OM such that $OM \cdot ON = r^2$.*

Recall that, when we studied the sphere, inversion also mapped the points O and ∞ to each other. We'll consider these two be inverses for the sake of simplicity.

Problem 3 *We say that two circles are orthogonal if their tangent lines are orthogonal at the points where they intersect. Show that a circle orthogonal to ω maps to itself under inversion with respect to ω.*

Problem 4 *Show that the inversion of any circle is a circle. (Here, we consider lines to be circles passing through ∞.)*

Problem 5 *In general, two shapes intersect at an angle θ if their tangent lines at that point intersect at an angle θ. Show that the inversion of any shapes that intersect at an angle θ still intersect at angle θ.*

Problem 6 *Define the cross ratio between any four points A, B, C, D by*

$$[A, B; C, D] = \frac{AC \cdot BD}{AD \cdot BC}$$

Show that, when we invert A, B, C, and D about a point that is neither of these four points, the cross ratio stays the same; that is, $[A, B; C, D] = [i_\omega(A), i_\omega(B); i_\omega(C), i_\omega(D)]$

3 The Hyperbolic Plane

Definition 4 *In Poincaré's model of the hyperbolic plane, also known as the **Poincaré disk**:*

- *The points are all points on the plane that are less than 1 away from the origin in distance.*
- *The points on the unit circle S^1 (exactly 1 away from the origin in distance) are called the **ideal points**.*
- *The (hyperbolic) lines are given by circles orthogonal to S^1. Note that, as before, straight lines are considered circles through ∞. Lines are **parallel** if they do not intersect.*

Problem 7 *Which straight lines are hyperbolic lines? Why?*

Definition 5 *Let A and B be points in the hyperbolic plane. Let P and Q be the ideal endpoints of the hyperbolic line connecting A and B such that P is closer to B and Q is closer to A. The* ***hyperbolic distance*** *from A to B is given by*

$$d(A, B) := \log([A, B; P, Q])$$

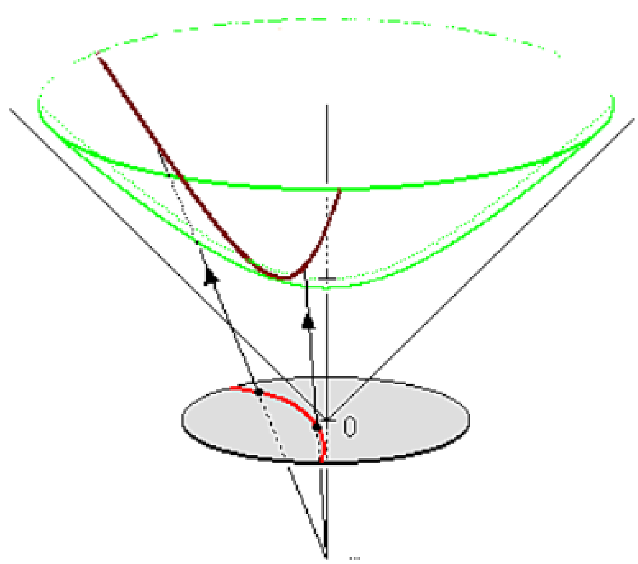
where the cross ratio uses the usual distance in the plane.

Problem 8 *We typically use base e for logarithms unless otherwise specified, but in the case of definition 5, it actually doesn't matter at all what the base is, because we can change the base. Show that*

$$\log_a(c) = log_a(b) \log_b(c)$$

Using the formula, explain why the base of the logarithm doesn't matter.

While we can think of Definitions 4 and 5 as abstract definitions of things called "Point", "Line", and "Distance", the Poincaré model is really a projection of a hyperbolic surface, as shown below. The distance as defined above is based on the distance on the surface, so we will be working mainly with the disk model as it is nicer to compute.



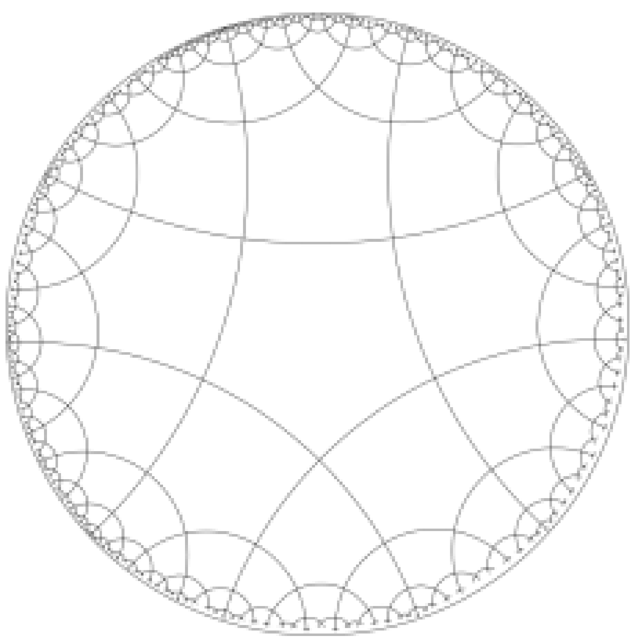
Problem 9 *Show that:*

- $d(A, B) \geq 0$ *with equality if and only if* $A = B$.

- $d(A, B) = d(B, A)$

- *If* $A, B,$ *and* C *are hyperbolic collinear (in that order),* $d(A, C) = d(A, B) + d(B, C)$.

In general, if three points are not collinear, we have the *hyperbolic triangle inequality*, which is a result analogous to the one for normal distance. Just like in Euclidean geometry, a triangle has three sides which are hyperbolic line segments, a quadrilateral four such sides, and so on. The following shows a tiling of the Poincaré disk using *hyperbolic pentagons*.



Problem 10 *We can see a lot of parallel lines in the previous image. Prove that the hyperbolic plane satisfies the second "alternative" parallel postulate. Given any line and a point not on that line, how many parallel lines are there passing through that point?*

Problem 11 *A* ***hyperbolic circle*** *is the set of points equidistant (with respect to the hyperbolic distance) from some point. What is the Euclidean shape of a hyperbolic circle centered at the origin? What about at any other point?*

4 Angles in the Hyperbolic Plane

In Euclidean geometry, the angles of a triangle add up to 180 degrees (and quadrilaterals 360 degrees, and so on). In hyperbolic geometry, we can still make sense of angles (as the angle between the tangent lines) but as we can see on the previous image, it seems to not be the case that pentagons have angles which add up to 540 degrees.

Problem 12 *Show that the angles of a (non-degenerate) triangle add up to less than* 180 *degrees. (Hint: Use an inversion to move one vertex to the origin and use Problem* 5*.)*

Definition 6 *Given a line L and a point P not on L, let R_1 and R_2 be the hyperbolic rays from P to the ideal endpoints of L. The **angle of parallelism** is the angle between R_1 and the perpendicular from P to L.*

Problem 13 *Show that the angle of parallelism also equals the angle between R_2 and the perpendicular from P to L.*

Like in Euclidean geometry, the length of this perpendicular from P to L is said to be the *distance from P to L*. Unlike in Euclidean geoemtry, however, we have a very easy way to find this distance in hyperbolic space:

Theorem 2 *(Lobachevsky's Theorem) Let L be a line and P a point not on that line with angle of parallelism θ. The distance d from P to L is given by*

$$d = -\log\left(\tan\left(\frac{\theta}{2}\right)\right)$$

Problem 14 *(Bonus) Prove Lobachevsky's Theorem.*