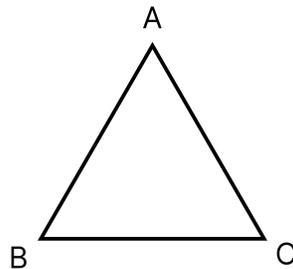


Symmetries and Groups

We've identified the algebraic axioms for the real numbers, and have even mentioned some number systems outside the real line that follow similar axioms. However, there are many other kinds of objects that satisfy these axioms. We first study objects which we can "add" in some way.

Rotation group of an equilateral triangle

Let's say we want to study the symmetries of the equilateral triangle below. To do so, we can start by first only studying the rotations of the triangle about its center.

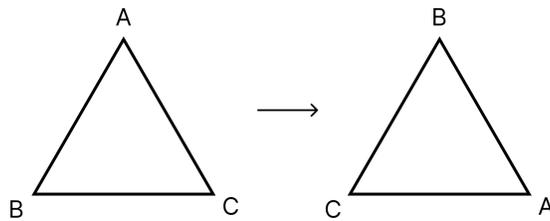


Problem 1. *Suppose I rotate the triangle by 840 degrees counterclockwise. What rotation less than 360 degrees clockwise achieves the same orientation of the triangle?*

Using the same tricks when solving Problem 1, we can study the rotational symmetry of the equilateral triangle by studying the following set.

Definition 1. *The rotation group of this equilateral triangle, written as R_Δ , is the set of all clockwise rotations less than 360 degrees about its center that send the vertices back on top of the vertices.*

For example, rotating the triangle by 120 degrees clockwise maps the vertices onto themselves, as below.



Hence, the rotation by 120 degrees clockwise is in the rotation group R_Δ . Let's write rotation by 120 degrees clockwise as r_{120} . Similarly, rotation by 0 degrees clockwise r_0 and rotation by 240 degrees clockwise r_{240} are also in the rotation group R_Δ . It turns out that r_0, r_{120}, r_{240} are the only rotations in the rotation group.

Problem 2. *Why are r_0, r_{120}, r_{240} the only objects in the rotation group?*

We can now define a natural way of adding the objects in the rotation group of the triangle.

Definition 2. *Given two rotations x, y in R_Δ , we define $y+x$ to be the unique rotation z in R_Δ that gives the same orientation of the triangle as given by doing x first and then y .*

Example 1. $r_{120} + r_0 = r_{120}$ since rotating clockwise by 0 degrees and then rotating clockwise by 120 degrees is the same as just rotating clockwise by 120 degrees.

Example 2. $r_{120} + r_{240} = r_0$ since rotating clockwise by 240 degrees and then rotating clockwise by 120 degrees is the same as just not rotating at all.

Problem 3. *Answer the following equalities.*

$$\begin{array}{lll}
 r_0 + r_0 = \underline{\hspace{2cm}} & r_0 + r_{120} = \underline{\hspace{2cm}} & r_0 + r_{240} = \underline{\hspace{2cm}} \\
 r_{120} + r_0 = \underline{\hspace{2cm}} & r_{120} + r_{120} = \underline{\hspace{2cm}} & r_{120} + r_{240} = \underline{\hspace{2cm}} \\
 r_{240} + r_0 = \underline{\hspace{2cm}} & r_{240} + r_{120} = \underline{\hspace{2cm}} & r_{240} + r_{240} = \underline{\hspace{2cm}}
 \end{array}$$

From Problem 3, we conclude that R_Δ with its definition of addition satisfies the following axioms.

Axiom 1 (Closure of addition). *For rotations x, y in R_Δ , we know that $x + y$ is also in R_Δ .*

Axiom 2 (Commutativity of addition). *For rotations x, y in R_Δ , we know that $x + y = y + x$.*

Axiom 3 (Identity of addition). *There is a rotation a in R_Δ such that, for any rotation x in R_Δ , we have $a + x = x + a = x$. We call a the additive identity.*

Problem 4. *Which number is a , the additive identity of R_Δ ?*

Axiom 4 (Inverse of addition). *For any rotation x in R_Δ , there is a corresponding rotation $-x$ in R_Δ such that $(-x) + x = x + (-x) = a$ where a is the additive identity of R_Δ . We call $-x$ the additive inverse of x .*

Problem 5. *Answer the following equalities.*

$$-r_0 = \underline{\hspace{2cm}} \qquad -r_{120} = \underline{\hspace{2cm}} \qquad -r_{240} = \underline{\hspace{2cm}}$$

If you check a little bit more, you can prove the following fact as well (which is obvious geometrically).

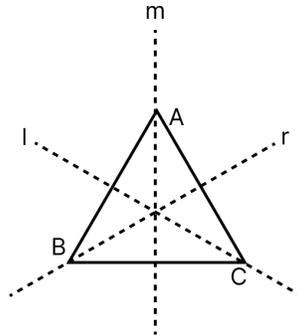
Axiom 5 (Associativity of addition). *For rotations x, y, z in R_Δ , we know that $x + (y + z) = (x + y) + z$. This lets us ignore parentheses when adding.*

Altogether, R_Δ is a set with a way of adding that satisfies all of the same addition axioms that the real numbers do!

Symmetry group of an equilateral triangle

We originally wanted to study all of the Euclidean symmetries of the equilateral triangle. Well, these must be combinations of rotations (about the center) and reflections (across axes passing through the triangle's center) that map vertices onto vertices. Note that only combinations of rotations and reflections are possible since these are the only transformations that preserve distances and angles (excluding translations, which do not map vertices onto vertices).

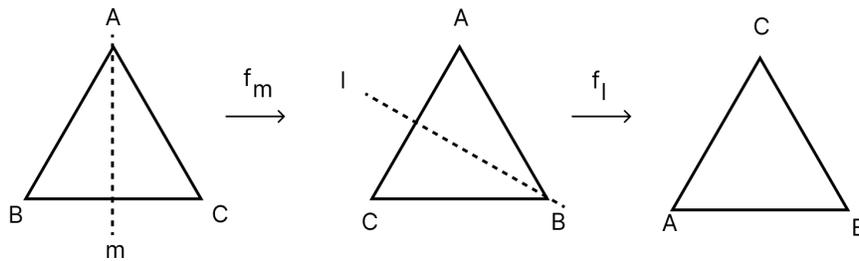
With a bit of geometric intuition, one can see that the only reflections of the triangle that map vertices onto vertices are f_l, f_m, f_r which are, respectively, the reflections across the lines l, m, r in the diagram below.



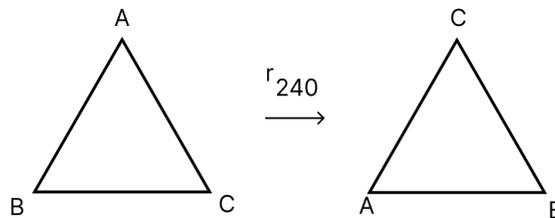
This observation lets us define the following.

Definition 3. Let F_Δ denote the set of f_l, f_m, f_r . The symmetry group of the equilateral triangle, written S_Δ , is the set of symmetries in R_Δ and the symmetries in F_Δ . Given two symmetries x, y in S_Δ , we define $y + x$ to be the unique symmetry z in S_Δ that gives the same orientation of the triangle as given by doing x first and then y .

Example 3. Let's see what $f_l + f_m$ does to a triangle with labeled vertices. Note that our definition of addition tells us we must do f_m first.



But is there a way to get from the left-most arrangement of vertices to the right-most using a symmetry in S_Δ ? Yes, there is!



Note that the rotations in S_Δ (and R_Δ) are all clockwise by definition. Altogether, we say that $f_l + f_m = r_{240}$.

We want to show that we can add any two symmetries in S_Δ to get another symmetry in S_Δ (meaning, we want to show addition is closed). We have already done this for R_Δ in Problem 3. So, we only have to do the following.

Problem 6. *Answer the following equalities by drawing triangles and seeing how vertices change under each sum of symmetries. Don't forget that all rotations are clockwise!*

$$f_l + f_m = \underline{\hspace{2cm}} \quad f_l + f_r = \underline{\hspace{2cm}} \quad f_m + f_l = \underline{\hspace{2cm}}$$

$$f_m + f_r = \underline{\hspace{2cm}} \quad f_r + f_l = \underline{\hspace{2cm}} \quad f_r + f_m = \underline{\hspace{2cm}}$$

$$r_0 + f_l = \underline{\hspace{2cm}} \quad r_0 + f_m = \underline{\hspace{2cm}} \quad r_0 + f_r = \underline{\hspace{2cm}}$$

$$f_l + r_0 = \underline{\hspace{2cm}} \quad f_m + r_0 = \underline{\hspace{2cm}} \quad f_r + r_0 = \underline{\hspace{2cm}}$$

This problem continues to the next page.

This is a continuation of the problem on the prior page.

$$r_{120} + f_l = \underline{\hspace{2cm}} \quad r_{120} + f_m = \underline{\hspace{2cm}} \quad r_{120} + f_r = \underline{\hspace{2cm}}$$

$$f_l + r_{120} = \underline{\hspace{2cm}} \quad f_m + r_{120} = \underline{\hspace{2cm}} \quad f_r + r_{120} = \underline{\hspace{2cm}}$$

$$r_{240} + f_l = \underline{\hspace{2cm}} \quad r_{240} + f_m = \underline{\hspace{2cm}} \quad r_{240} + f_r = \underline{\hspace{2cm}}$$

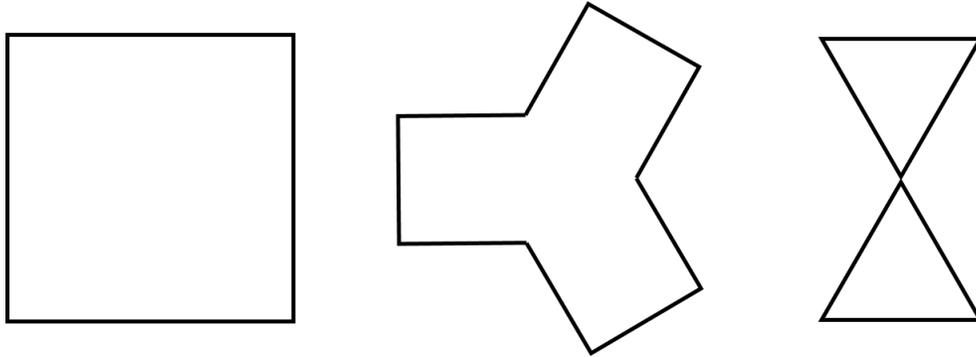
$$f_l + r_{240} = \underline{\hspace{2cm}} \quad f_m + r_{240} = \underline{\hspace{2cm}} \quad f_r + r_{240} = \underline{\hspace{2cm}}$$

Again, geometric intuition (supported by many computations) does show that addition in S_Δ satisfies associativity. In turn, Problem 3 along with Problem 6 imply that addition in S_Δ satisfies four out of all five addition axioms.

Problem 7. *Which addition axiom is not satisfied in S_Δ ?*

Symmetric groups of other shapes

In class, the instructors will pass out cutouts of the following three shapes that we will call the square, the clover, and the hourglass (from left to right). You will spend the rest of class playing around with these shapes to better understand their symmetries! The instructors will help guide you through the following problems.



You can define the symmetry groups of each of these shapes in the same way as we did with the equilateral triangle: the set of the vertex-preserving rotations and reflections.

Problem 8. *Complete the following parts.*

(i) What are the clockwise rotation symmetries of the square?

(ii) What are the clockwise rotation symmetries of the clover?

(iii) What are the clockwise rotation symmetries of the hourglass?

Problem 9. Complete the following parts. In your answers, feel free to draw the shapes and any lines that are the axes of reflection (you should probably label the lines, like how we did for the equilateral triangle).

(i) What are the reflection symmetries of the square?

(ii) What are the reflection symmetries of the clover?

(iii) What are the reflection symmetries of the hourglass?

We can also define addition in the symmetry groups in the same way that we did for the equilateral triangle. After playing around with these shapes, you will see that addition in each of these symmetry groups satisfies the closure, associativity, identity, and inverse axioms.

Problem 10. Exactly one symmetry group for the three shapes above has addition that satisfies the commutativity axiom. Which is it? Hint: Label the vertices of your cutouts and calculate symmetry additions.