Axiom Reference Sheet

It can be really tough to remember all of the algebraic axioms at first. To help you out, here is a reference sheet listing them all! We also include a table showing which types of numbers on the real line satisfy which axioms.

Let $S$ be a set of objects. Suppose we can add and multiply together the objects in $S$, meaning there are binary operations $+$ and $\cdot$ on $S$. Then, we have the following definitions of algebraic axioms on $S$ (we put formal and casual versions to help you remember).

**Axiom 1** (Closure of addition). *We say that $S$ is closed under addition if the following is true.*

- Formally: $x + y$ is in $S$ whenever $x, y$ are in $S$.
- Casually: addition keeps you inside of $S$.

**Axiom 2** (Closure of multiplication). *We say that $S$ is closed under multiplication if the following is true.*

- Formally: $x \cdot y$ is in $S$ whenever $x, y$ are in $S$.
- Casually: multiplication keeps you inside of $S$.

**Axiom 3** (Associativity of addition). *We say that $S$ satisfies the associativity of addition if the following is true.*

- Formally: $x + (y + z) = (x + y) + z$ for objects $x, y, z$ in $S$.
- Casually: we can ignore parentheses when adding.

**Axiom 4** (Associativity of multiplication). *We say that $S$ satisfies the associativity of multiplication if the following is true.*

- Formally: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for objects $x, y, z$ in $S$.
- Casually: we can ignore parentheses when multiplying.
Axiom 5 (Commutativity of addition). We say that $S$ satisfies the commutativity of addition if the following is true.

- Formally: $x + y = y + x$ for objects $x, y$ in $S$.
- Casually: we can ignore order when adding.

Axiom 6 (Commutativity of multiplication). We say that $S$ satisfies the commutativity of multiplication if the following is true.

- Formally: $x \cdot y = y \cdot x$ for objects $x, y$ in $S$.
- Casually: we can ignore order when multiplying.

Axiom 7 (Identity of addition). We say that $S$ has an additive identity $a$ if the following is true.

- Formally: There is an object $a$ in $S$ such that $a + x = x + a = x$ for all objects $x$ in $S$.
- Casually: There’s an object that does nothing when adding with it.

Axiom 8 (Identity of multiplication). We say that $S$ has a multiplicative identity $m$ if the following is true.

- Formally: There is an object $m$ in $S$ such that $m \cdot x = x \cdot m = x$ for all objects $x$ in $S$.
- Casually: There’s an object that does nothing when multiplying by it.
Axiom 9 (Inverse of addition). Suppose $S$ has an additive identity $a$. We say that $S$ satisfies the additive inverse axiom if the following is true.

- Formally: For an object $x$ in $S$, there is a corresponding object $y$ in $S$ such that $y + x = x + y = a$. We call $y$ the additive inverse of $x$.
- Casually: There’s always an opposite object when adding.

Axiom 10 (Inverse of multiplication). Suppose $S$ has a multiplicative identity $m$. We say that $S$ satisfies the multiplicative inverse axiom if the following is true.

- Formally: For an object $x$ in $S$ (not equal to the additive identity $a$ if it exists in $S$), there is a corresponding object $y$ in $S$ such that $y \cdot x = x \cdot y = m$. We call $y$ the multiplicative inverse of $x$.
- Casually: There’s always an opposite object when multiplying (except for the additive identity $a$ if it exists in $S$).

Axiom 11 (Left distributivity). We say that $S$ satisfies left distributivity if the following is true.

- Formally: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for objects $x, y, z$ in $S$.
- Casually: Multiplication jumps over addition on the left.

Axiom 12 (Right distributivity). We say that $S$ satisfies right distributivity if the following is true.

- Formally: $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ for objects $x, y, z$ in $S$.
- Casually: Multiplication jumps over addition on the right.
Recall the types of numbers on the real line from the second handout (What is a Number Part II): naturals \( \mathbb{N} \), wholes \( \mathbb{W} \), integers \( \mathbb{Z} \), rationals \( \mathbb{Q} \), reals \( \mathbb{R} \). We also had an extra credit discussion about the irrationals \( \mathbb{P} \) which are defined to be all the real numbers that are not rational. Below is a chart of which number type satisfies the above axioms. A checkmark \( \checkmark \) means the axiom is satisfied while an \( \times \) means it is not.

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