Hyperbolic Geometry I

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ORMC Advanced 1 - Winter 2023

1 Non-Euclidean Geometries

Euclid's *Elements*, often considered the first mathematical treatise, begins with axioms that are used to prove theorems that are familiar to us about shapes in the plane. One of Euclid's axioms was as follows. Although Euclid himself stated it differently, it is equivalent to the following formulation:

Definition 1 *The **parallel postulate** states that given any line L and any point P not on L, there exists a unique line through P parallel to L.*

Euclidean geometry describes the *plane*, which can be expressed as the coordinate plane $\mathbb{R}^2$ - that is, ordered pairs of real numbers.

Problem 1 *Show that the coordinate plane $\mathbb{R}^2$ satisfies the parallel postulate. (Hint: Write lines in point-slope form. When are they parallel?)*

Solution: First, if a line is vertical, it is of the point $x = a$. Any point not on $x = a$ has x-coordinate b different from a, and there is a unique vertical line through that point (namely, $x = b$). Now if a line is not vertical, it can be written as $y = mx + b$. If any other line has slope $n \neq m$, we can solve the equation $mx + b = nx + c$ for x, so the two lines must intersect and cannot be parallel, so given any point (x_0, y_0) not on $y = mx + b$ there is a unique line, namely $y - y_0 = m(x - x_0)$, parallel to $y = mx + b$.

When studying *non-Euclidean geometry*, while we could negate any of Euclid's axioms, the most interesting examples occur when we negate the parallel postulate. The negation of "there exists a unique" is "there does not exist or there exists more than one", so we have two options for replacement axioms:

Definition 2 *The two alternative parallel postulates are:*

- *No two lines are parallel.*
- *Given any line L and point P not lying on L, there are at least two lines through P parallel to L.*

This week we will study one particular example satisfying the second of these.

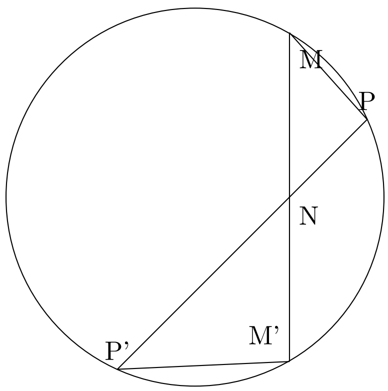
2 Circle Inversions

We begin with a useful theorem.

Theorem 1 *(Power of a Point): Let* ω *be a circle,* N *a point not on* ω*, and* L *any line through which intersects* ω *at two points* M *and* M'*. (Note: If* L *is tangent to* ω*, we consider the tangency point to be both* M *and* M'*) Then the value* $NM \cdot NM'$ *is the same for any choice of line* L*.*

Problem 2 *(Bonus) Prove Theorem 1.*

Solution: The below shows the case where N is inside the circle; the case where N is outside is similar. The triangles NMP and $NP'M'$ are similar because angles NMP and $NP'M'$ subtend the same arc.

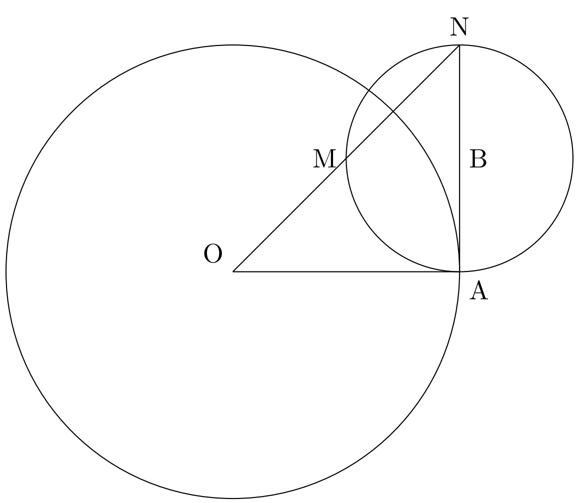


Definition 3 *Let* ω *be a circle with center* O *and radius* r*. The inversion of any point* $M \neq O$ *in the plane (denoted* $i_\omega(M)$*) is the point* N *lying on ray* OM *such that* $OM \cdot ON = r^2$*.*

Recall that, when we studied the sphere, inversion also mapped the points O and ∞ to each other. We'll consider these two be inverses for the sake of simplicity.

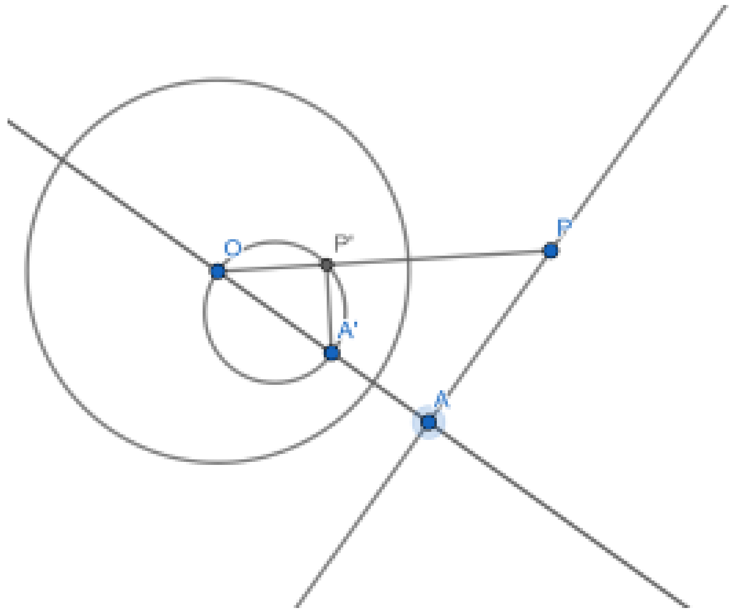
Problem 3 *We say that two circles are orthogonal if their tangent lines are orthogonal at the points where they intersect. Show that a circle orthogonal to* ω *maps to itself under inversion with respect to* ω*.*

Solution: Consider the following circle centered at B which is orthogonal to the one centered at O. Given any point M on the former circle, drawing the ray OM until it intersects the circle again at N, we see that $r^2 = OA^2 = OM \cdot ON$ by Power of a Point.



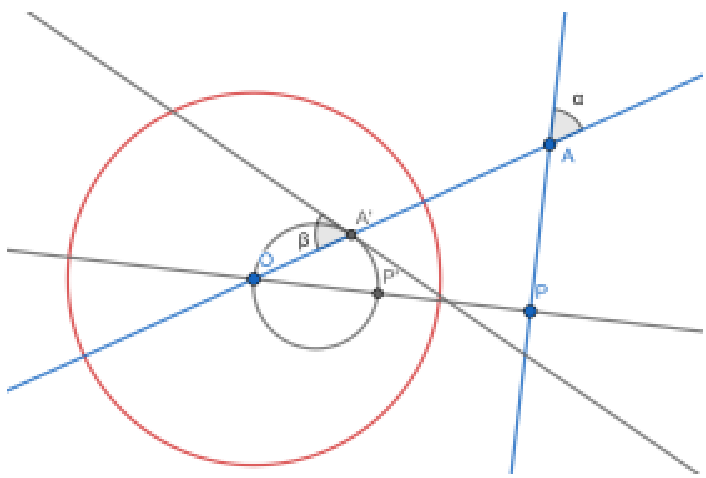
Problem 4 *Show that the inversion of any circle is a circle. (Here, we consider lines to be circles passing through* ∞*.)*

Solution: The following shows the case of a line; a circle follows similarly.



Problem 5 *In general, two shapes intersect at an angle* θ *if their tangent lines at that point intersect at an angle* θ*. Show that the inversion of any shapes that intersect at an angle* θ *still intersect at angle* θ*.*

Solution: Assume that the two shapes intersecting are lines (if not, just replace them with their tangent lines.



Problem 6 *Define the cross ratio between any four points A, B, C, D by*

$$[A, B; C, D] = \frac{AC \cdot BD}{AD \cdot BC}$$

Show that, when we invert A, B, C, and D about a point that is neither of these four points, the cross ratio stays the same; that is, $[A, B; C, D] = [i_\omega(A), i_\omega(B); i_\omega(C), i_\omega(D)$

Solution: If A and C are collinear, this follows easily. If they are not, then triangle OAC is similar to triangle $Oi_\omega(A)i_\omega(C)$, and similarly for each other segment, so that

$$\frac{AC \cdot i_\omega(B)i_\omega(C) \cdot BD \cdot i_\omega(A)i_\omega(D)}{BC \cdot i_\omega(A)i_\omega(C) \cdot AD \cdot i_\omega(B)i_\omega(D)} = 1$$

3 The Hyperbolic Plane

Definition 4 *In Poincaré's model of the hyperbolic plane, also known as the* ***Poincaré disk****:*

- *The points are all points on the plane that are less than 1 away from the origin in distance.*
- *The points on the unit circle S^1 (exactly 1 away from the origin in distance) are called the* ***ideal points****.*
- *The (hyperbolic) lines are given by circles orthogonal to S^1. Note that, as before, straight lines are considered circles through ∞. Lines are* ***parallel*** *if they do not intersect.*

Problem 7 *Which straight lines are hyperbolic lines?*

Solution: A straight line is orthogonal to S^1 if and only if it is a diameter of S^1.

Definition 5 *Let A and B be points in the hyperbolic plane. Let P and Q be the ideal endpoints of the hyperbolic line connecting A and B such that P is closer to B and Q is closer to A. The* ***hyperbolic distance*** *from A to B is given by*

$$d(A,B) := \log([A,B;P,Q])$$

where the cross ratio uses the usual distance in the plane.

Problem 8 *We typically use base e for logarithms unless otherwise specified, but in the case of definition 5, it actually doesn't matter at all what the base is, because we can change the base. Show that*

$$\log_a(c) = log_a(b)\log_b(c)$$

Using the formula, explain why the base of the logarithm doesn't matter.

Solution: An intuitive proof of the change of base formula is as follows: the number of powers of a to make c is the same as the number of powers of a to make c, if we first take powers of a to make b. A formal proof can be given by exponentiating. So given any base of the logarithm, we can change it to any other base by multiplying by $\log_a(b)$.

While we can think of Definitions 4 and 5 as abstract definitions of things called "Point", "Line", and "Distance", the Poincaré model is really a projection of a hyperbolic surface, as shown below. The distance as defined above is based on the distance on the surface, so we will be working mainly with the disk model as it is nicer to compute.

Problem 9 *Show that:*

- $d(A,B) \geq 0$ *with equality if and only if* $A = B$.

  Solution: $\log(x) \geq 0$ *if and only if* $x \geq 1$, *with equality if and only if* $x = 1$, *so we look at the cross ratio. Since* $AP \geq BP$ *and* $BQ \geq AQ$, *the numerator of the cross ratio is greater than the denominator, unless* $A = B$ *in which case both of these are equal.*

- $d(A,B) = d(B,A)$

  Solution: Switching A and B also switches P and Q, so looking at the cross ratio,

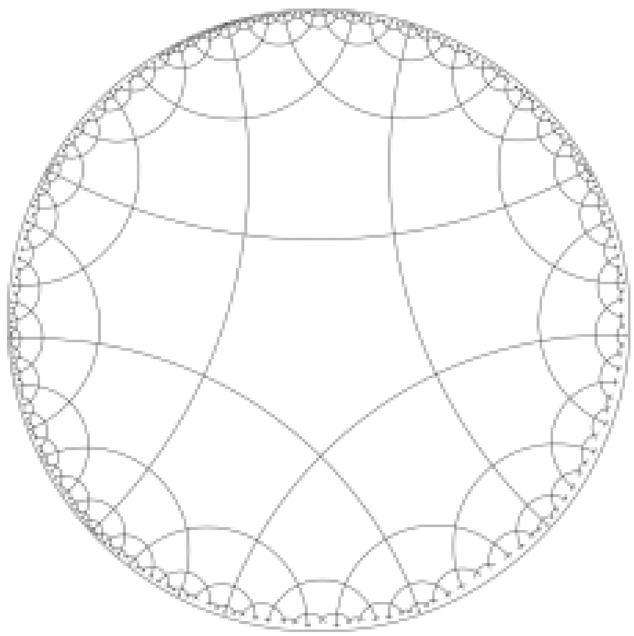
$$\frac{AP \cdot BQ}{AQ \cdot BP} = \frac{BQ \cdot AP}{BP \cdot AQ}$$

- *If $A, B,$ and C are hyperbolic collinear (in that order), $d(A,C) = d(A,B) + d(B,C)$.*

  Solution: If $A, B,$ and C are hyperbolic collinear, extend that hyperbolic line to P and Q. Without loss of generality, suppose the points are Q, A, B, C, P in that order (otherwise, relabel P, Q). Then

$$\begin{aligned} d(A,B) + d(B,C) &= \log\left(\frac{AP \cdot BQ}{AQ \cdot BP}\right) + \log\left(\frac{BP \cdot CQ}{BQ \cdot CP}\right) \\ &= \log\left(\frac{AP \cdot BQ}{AQ \cdot BP} \times \frac{BP \cdot CQ}{BQ \cdot CP}\right) = \log\left(\frac{AP \cdot CQ}{AQ \cdot CP}\right) = d(A,C) \end{aligned}$$

In general, if three points are not collinear, we have the *hyperbolic triangle inequality*, which is a result analogous to the one for normal distance. Just like in Euclidean geometry, a triangle has three sides which are hyperbolic line segments, a quadrilateral four such sides, and so on. The following shows a tiling of the Poincaré disk using *hyperbolic pentagons*.



Problem 10 *We can see a lot of parallel lines in the above image. Prove that the hyperbolic plane satisfies the second "alternative" parallel postulate above. Given any line and a point not on that line, how many parallel lines are there passing through that point?*

Solution: Various different proofs. There are infinitely many different parallel lines passing through that point.

Problem 11 *A **hyperbolic circle** is the set of points equidistant (with respect to the hyperbolic distance) from some point. What is the Euclidean shape of a hyperbolic circle centered at the origin? What about at any other point?*

Solution: The Euclidean shape of any hyperbolic circle centered at the origin is a circle centered at the origin. To see this, consider the distance between the origin and any point A. The points P and Q given in Definition 5 lie at the ends of the diameter through A, so that the hyperbolic distance from A to the origin depends only on the Euclidean distance from A to the origin, so a hyperbolic circle centered at the origin is the set of points with the same Euclidean distance to the origin, and therefore is a Euclidean circle.

A hyperbolic circle about any other point is still a Euclidean circle, but not centered at that point. This is because inversion preserves lengths, so we can switch any other point for the origin, and invert back to get a circle by Problem 4. But inversion doesn't preserve the center of a circle in general.

4 Angles in the Hyperbolic Plane

In Euclidean geometry, the angles of a triangle add up to 180 degrees (and quadrilaterals 360 degrees, and so on). In hyperbolic geometry, we can still make sense of angles (as the angle between the tangent lines) but as we can see on the previous image, it seems to not be the case that pentagons have angles which add up to 540 degrees.

Problem 12 *Show that the angles of a (non-degenerate) triangle add up to less than* 180 *degrees. (Hint: Use an inversion to move one vertex to the origin and use Problem* 5*.)*

Solution: By inverting about the right circle, assume one vertex is at the origin. Then two edges are straight (they are along diameters) so the hyperbolic triangle is immediately comparable to the Euclidean triangle with those vertices. Since the hyperbolic triangle must be concave, its angles must be strictly smaller.

Definition 6 *Given a line L and a point P not on L, let R_1 and R_2 be the hyperbolic rays from P to the ideal endpoints of L. The* ***angle of parallelism*** *is the angle between R_1 and the perpendicular from P to L.*

Problem 13 *Show that the angle of parallelism also equals the angle between R_2 and the perpendicular from P to L.*

Solution: Essentially by symmetry.

Like in Euclidean geometry, the length of this perpendicular from P to L is said to be the *distance from P to L*. Unlike in Euclidean geoemtry, however, we have a very easy way to find this distance in hyperbolic space:

Theorem 2 *(Lobachevsky's Theorem) Let L be a line and P a point not on that line with angle of parallelism θ. The distance d from P to L is given by*

$$d = -\log\left(\tan\left(\frac{\theta}{2}\right)\right)$$

Problem 14 *(Bonus) Prove Lobachevsky's Theorem.*