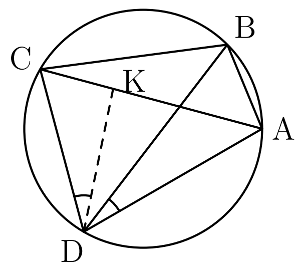
ORMC AMC 10/12 Group
Inequalities

January 22, 2023

1 Cyclic Quadrilateral Theorems

1.1 Ptolemy's Theorem



As shown in the diagram above, begin by drawing the segment $\overline{DK}$ such that $\angle CDK \cong \angle BDA$. Note that we then also have $\angle CDB \cong \angle KDA$.

Also, by the inscribed angle theorem, we have $\angle DCK \cong \angle DBA$, and $\angle DBC \cong \angle DAK$.

So, we have 2 pairs of similar triangles: $\triangle DKC \sim \triangle DAB$ and $\triangle DBC \sim \triangle DAK$

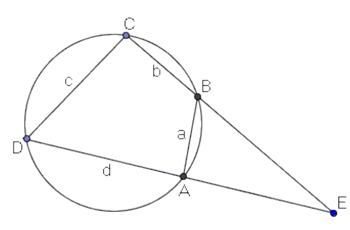
These give us the following:

$$\frac{CD}{KC} = \frac{BD}{AB} \implies AB \cdot CD = BD \cdot KC, \quad \frac{BC}{BD} = \frac{AK}{AD} \implies BC \cdot AD = BD \cdot AK$$

Adding these together gives us ptolemy's theorem:

$$\boxed{AB \cdot CD + BC \cdot AD = BD \cdot AC.}$$

1.2 Brahmagupta's Formula



We prove the case where some pair of opposite sides is not parallel. The other case is left as an exercise. Extend the non-parallel sides to meet at a point E, as shown above. Then, let $x = DE$ and $y = CE$. By Heron's Formula, $\triangle CDE$ has area $\frac{1}{4}\sqrt{(x+y+c)(x+y-c)(x-y+c)(-x+y+c)}$.

The area of triangle ABE is $\frac{a^2}{c^2}$ of this, so the cyclic quadrilateral's area is $1 - \frac{a^2}{c^2}$ of it. So, all we have to do is rewrite each of the terms in Heron's formula in terms of a, b, c, d, using the fact that $\frac{b}{x} = \frac{d}{y} = \frac{c-a}{c}$. This follows from the similarity of ABE and CDE, which we showed when proving power of a point. We get the following:

$$x+y+c = c\frac{b+d}{c-a}+c = c\frac{b+d+c-a}{c-a} = 2c\frac{s-a}{c-a}, \quad x+y-c = c\frac{b+d}{c-a}-c = c\frac{b+d-c+a}{c-a} = 2c\frac{s-c}{c-a},$$
$$x-y+c = c\frac{b-d}{c+a}+c = c\frac{b-d+c+a}{c+a} = 2c\frac{s-d}{c+a}, \quad -x+y+c = c\frac{-b+d}{c+a}+c = c\frac{-b+d+c+a}{c+a} = 2c\frac{s-b}{c+a}$$

Plugging in, we find that the area of the quadrilateral is:

$$[ABCD] = \frac{c^2-a^2}{c^2}\frac{1}{4}\frac{4c^2}{c^2-a^2}\sqrt{(s-a)(s-c)(s-b)(s-d)} = \boxed{\sqrt{(s-a)(s-b)(s-c)(s-d)}.}$$

1.3 Examples

1. **(2022 AMC 10A #15)** Quadrilateral $ABCD$ with sides $AB = 7, BC = 24, CD = 20, DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where $a, b,$ and c are positive integers such that a and c have no common prime factor. What is $a+b+c$?

2. **(2016 AMC 10A #24)** A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

1.4 Exercises

1. **(Brahmagupta's Formula, Case 2)** Show that Brahmagupta's formula still holds when both pairs of opposite sides are parallel. (This is very simple– don't overthink it!)

2. **(2018 AMC 12A #20)** Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a+b+c$?

3. **(1991 AIME #14)** A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

4. **(Pre-2005 Mock AIME 3 #7)** $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1, CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p+q$.

2 Inequalities

Review of some inequality basics:

- Inequalities are transitive: $a < b, b < c \implies a < c$
- You can add (subtract) from both sides of an inequality: $a < b \iff a + c < b + c$.
- You can multiply (divide) both sides by a positive value: $c > 0, a < b \implies ca < cb$.
- Multiplying (dividing) by a negative value "flips" the inequality: $c < 0, a < b \implies ca > cb$.
  - One way to see this is to think of it as multiplying by $|c|$, then subtracting the values from both sides:

$$c < 0, a < b \implies |c|a < |c|b \implies |c|a - |c|b < 0 \implies cb = -|c|b < -|c|a = ca$$

Note that due to the last point, it is important to be careful when multiplying or dividing an inequality by a variable that could be negative!

Solutions are often in *interval notation*. An interval is written like $(a, b), [a, b], [a, b)$, or $(a, b]$. This indicates to include all points between a and b. A parenthesis indicates to exclude the boundary point, while a square bracket indicates to include it.

2.1 Tips and Strategies

There are many ways to approach solving inequalities; here are some tips:

- Start by testing some values (like 0, 1, etc) just to get a feel for where an inequality might be true or false.
- Bring all the terms to one side, to make the inequality a statement about whether the expression is positive or negative. This is generally easier to deal with.
- Treat the inequality as an equation. Finding roots (or points where it is undefined) is equivalent to finding *"pivot points"*. An inequality is uniformly true or false between its pivot points, so you only need to check one value in each interval between pivot points.

2.2 Examples

1. How many integers satisfy $|x| + 1 \geq 3$ and $|x - 1| < 3$?

2. Find all $x \in \mathbb{R}$ satisfying $\frac{x-8}{x+5} + 4 \geq 3$.

2.3 Exercises

1. Solve $\frac{|x^2-81|}{x^2-36x} < 0$.

2. **(2010 AMC 10A #11)** The length of the interval of solutions of the inequality $a \leq 2x + 3 \leq b$ is 10. What is $b - a$?

3. **(2008 AMC 12A #14)** What is the area of the region defined by the inequality $|3x - 18| + |2y + 7| \leq 3$?

4. **(1998 AIME #2)** Find the number of ordered pairs (x, y) of positive integers that satisfy $x \leq 2y \leq 60$ and $y \leq 2x \leq 60$.

5. **(1992 AIME #3)** A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly .500. During the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than .503. What's the largest number of matches she could've won before the weekend began?

3 AM-GM

The AM-GM inequality is an important inequality that is frequently found on the AMC. Most often, you will see the two-variable case:

$$\frac{x+y}{2} \geq \sqrt{xy}, \quad \forall\, x, y \geq 0$$

The inequality above states that the arithmetic mean (AM) of two nonnegative variables is greater than or equal to the geometric mean of those variables. This comes from the "trivial inequality", which states that $x^2 \geq 0$, $\forall x \in \mathbb{R}$. In particular, we can "work backwards" towards the trivial inequality:

$$\frac{x+y}{2} \geq \sqrt{xy} \iff x+y \geq 2\sqrt{xy} \iff x+y-2\sqrt{xy} \geq 0$$

This is where it becomes important that $x, y \geq 0$, so that $x = \sqrt{x}^2, y = \sqrt{y}^2$. Hopefully you recognize the form above as a squared binomial:

$$x+y-2\sqrt{xy} = (\sqrt{x}-\sqrt{y})^2$$

Clearly this value is nonnegative (by the trivial inequality), so the AM-GM inequality follows.

It is also important to know the general case, for n variables:

$$\frac{x_1+x_2+\cdots+x_n}{n} = \sqrt[n]{x_1x_2\cdots x_n}$$

We can show this using a special type of induction, which involves showing that:

1. If the statement is true for n, then it is also true for $2n$.
2. If the statement is true for $n+1$, then it is also true for n.

Proof. 1. The first part uses the base case ($n = 2$). Suppose the statement is true for some $k \in \mathbb{N}$. Then, we have:

$$\frac{x_1+\cdots+x_k}{k} \leq \sqrt[k]{x_1\cdots x_k}, \quad \frac{x_{k+1}+\cdots+x_{2k}}{k} \leq \sqrt[k]{x_{k+1}\cdots x_{2k}}$$

for any nonnegative $x_1, \cdots, x_{2k}$. By the $n = 2$ case,

$$\frac{x_1+\cdots+x_{2k}}{2k} = \frac{\frac{x_1+\cdots+x_k}{k}+\frac{x_{k+1}+\cdots+x_{2k}}{k}}{2} \leq \sqrt{\sqrt[k]{x_1\cdots x_k}\sqrt[k]{x_{k+1}\cdots x_{2k}}} = \sqrt[2k]{x_1\cdots x_{2k}}.$$

So if the statement is true for $n = k$, then it is also true for $n = 2k$.

2. The second part involves showing that $n-1$ is actually a special case of n. Suppose that the statement is true for some $k > 1$. Then, consider the values: $x_1, x_2, \ldots, x_{k-1}, x_k$, where $x_k = \sqrt[k-1]{x_1x_2\cdots x_{k-1}}$. We have:

$$\frac{x_1+\cdots+x_{k-1}+x_k}{k} \geq \sqrt[k]{x_1x_2\cdots x_{k-1}x_k} = x_k^{(k-1)/k}\sqrt[k]{x_k} = x_k$$

$$\implies \frac{x_1+\cdots+x_{k-1}}{k} \geq x_k - \frac{1}{k}x_k = \frac{k-1}{k}x_k \implies \frac{x_1+\cdots+x_{k-1}}{k-1} \geq \sqrt[k-1]{x_1x_2\cdots x_{k-1}}.$$

So, the statement is true for all n, by induction. $\square$

Important note: For AM-GM, equality is achieved if and only if all of the x_i's are equal.

For those of you who don't know calculus, AM-GM tends to be a good way to solve "find the maximum/minimum" type problems. And for those of you who do know calculus, AM-GM can be a quicker way to find a maximum or minimum (compared to taking a derivative, solving, then plugging back in) where it is applicable.

3.1 Examples

1. Find the maximum of $2 - a - \frac{1}{2a}$ for all positive a.

2. Let $a > 5$. Find the smallest possible value of $a + \frac{4}{a-5}$.

3.2 Exercises

1. Suppose that $(a+1)(b+1)(c+1) = 8$ and $a, b, c \geq 0$. Show that $abc \leq 1$.

2. **(1983 AIME #9)** Find the minimum value of $\frac{9x^2 \sin^2 x + 4}{x \sin x}$ for $0 < x < \pi$.

3. **(2000 AMC 12 #12)** Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + A \cdot C$?

4. A ball is thrown upward from the top of a tower. If its height at time t is described by $-t^2 + 60t + 700$, what is the greatest height the ball reaches?

5. **(2020 AMC 12B #22)** What is the maximum value of $\frac{(2^t - 3t)t}{4^t}$ for real values of t?

6. **(USAMTS 31/1/3)** Circle ω is inscribed in unit square $PLUM$, and points I and E lie on ω such that U, I, and E are collinear. Find, the greatest possible area for $\triangle PIE$

7. (Hard) If the polynomial $x^3 - 12x^2 + ax - 64$ has only real, nonnegative roots, find a.