ORMC Olympiad Group
Winter: Week 3
Analysis: Functions and Polynomials

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Problems

1. **(IMOmath P3)** Function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x + f(x)$ for all $x \in \mathbb{R}$. Find all solutions of the equation $f(f(x)) = 0$
2. **(IMOmath P1)** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$.
3. **(TNMO-FR 2009)** Define $f(x) = \frac{x^5}{5x^4-10x^3+10x^2-5x+1}$ and $x_i = \frac{i}{2009}$. Compute
$$\sum_{i=1}^{2009} f(x_i)$$
4. **(Turkish JNMO 2000 Second Round P3)** $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation
$$f(x)f(y) - af(xy) = x + y$$
for every real numbers x, y. Find all possible real values of a.
5. **(AHSME 1975-modified)** Suppose $f(x)$ is a function defined for all real numbers x; so that $f(x) > 0$ for all x, and $f(a)f(b) = f(a+b)$ for all a and b. Among all such functions, determine if the following statement is always true.

$$f(b) \geq f(a) \text{ for all } b > a$$

6. **(Turkish NMO 2004 Second Round P4)** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$

7. **(IMO 1977)** Let $f(n)$ be a function defined on the set of all positive integers and with all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then $f(n) = n$ for each n.

8. **(Turkish NMO 2008 Second Round P4)** $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ satisfy the given conditions

$a)$ $f(0,0) = 1$, $f(0,1) = 1$,

$b)$ $\forall k \notin \{0,1\}$ $f(0,k) = 0$ and

$c)$ $\forall n \geq 1$ and k , $f(n,k) = f(n-1,k) + f(n-1,k-2n)$

find the sum $\displaystyle\sum_{k=0}^{\binom{2009}{2}} f(2008,k)$

9. Let $f(x) = x^2 - ax + 2020$ where a is a real number. Find a if $f(2020) = f(1048)$.

10. Let $f(x) = x^3 - 4039x^2 + Nx + 1$ where N is an integer. Find the remainder of N when divided by 1000 if $f(2020) = f(2019)$.

11. Let $f(x) = x^2 - 1$ and $g(x) = x - 1$. Find the sum of integers n which does not satisfy

$$(f(g(n))) > g(n-1)$$

12. (HMMT 2005 General 2) Find three real numbers $a < b < c$ satisfying:

$$\begin{aligned} a+b+c &= \frac{21}{4} \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} &= \frac{21}{4} \\ abc &= 1 \end{aligned}$$

13. The polynomial P satisfies $P(1) = 3$, $P(3) = 7$. Find the remainder of the polynomial when divided by $x^2 - 4x + 3$

14. (TNMO-FR 1998) Find the number of primes p, such that $x^3 - 5x^2 - 22x + 56 \equiv 0 \pmod{p}$ has no three distinct integer roots in $[0, p)$

15. Polynomial $P(x) = a_{2020}x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0$ satisfies $P(n) = 2^n$ for $n = -1000, -999, \ldots, 999, 1000$. Compute the sum of positive even indexed terms, ie compute $a_{2020} + a_{2018} + \cdots + a_2$

16. Find the monic polynomial with least degree which makes

$$(x-1)(x^2-1)(x^3-1)Q(x) \geq 0$$

for all $x \in \mathbb{R}$

17. Let $Q(x)$ be the monic polynomial with least degree possible which makes

$$(x^3 - 5x^2 + x - 5)(x^2 - 7x + 6)Q(x) \geq 0$$

for all $x > 4$. What is $Q(10)$?