What Is a Number? (Part II)

After having reviewed the class's answers to Lesson 1, we provide this handout as a reference and summary of what we've learned.

Types of numbers on the real line

Below is a diagram showing the standard types of numbers on the one-dimensional number line.

Real Numbers - $\mathbb{R}$: $\pi,\ e,\ -\sqrt{2},\ -1.23456789101112...$

Can represent real-world quantities that change continuously such as lengths, areas, or volumes.

Rational Numbers - $\mathbb{Q}$: $\frac{1}{2},\ \frac{-22}{7},\ 12.9782,\ -1.3333...$

Represent proportions and ratios between real-world objects.

Integers - $\mathbb{Z}$: $-1,\ -2,\ -100,\ -193782$

Can represent gains and losses of real-world objects.

Whole Numbers - $\mathbb{W}$: 0

Very similar to naturals, but can also represent absence of real-world objects.

Natural Numbers - $\mathbb{N}$: $1,\ 2,\ 100,\ 193782$

Can be used to count and order real-world objects

Problem 1. *Write any other real-world uses for the various types of real numbers in the diagram on the previous page.*

There are many (many) more types of "secret" numbers beyond the real number line. For example, how many of you have heard about complex numbers or quaternions? These "secret" numbers were created by mathematicians to simplify problems with complicated structures.

A problem we run into with these "secret" numbers is that they're not particularly natural. Adding and multiplying real numbers is simple, and their properties are obvious. But if we want to work with "secret" numbers in meaningful ways, we need to make sure they obey the same rules as our natural constructions.

Instead of tying the definition of all numbers down to their real-world applications, can we define rules that abstractly capture the properties we want? If so, we can discover many different mathematical objects which act quite similarly to real numbers. As long as they follow the same rules, why not call these objects "numbers" too?

Algebraic axioms of the real numbers

Remember that a binary operation is a way of combining two numbers into one. The two most important (and simple) binary operations are addition and multiplication. We are also interested in their *inverses* of subtraction and division, respectively. Here we list the various algebraic properties/laws that addition and multiplication satisfy for the real numbers.

Axiom 1 (Closure of addition)**.** *For real numbers x, y, we know that $x + y$ is also a real number.*

Axiom 2 (Closure of multiplication)**.** *For real numbers x, y, we know that $x \cdot y$ is also a real number.*

Problem 2. *Write in your own (casual) words what the word "closure" means in math for a binary operator.*

Axiom 3 (Associativity of addition)**.** *For real numbers x, y, z, we know that $x + (y + z) = (x + y) + z$. This lets us ignore parentheses when adding.*

Axiom 4 (Associativity of multiplication)**.** *For real numbers x, y, z, we know that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. This lets us ignore parentheses when multiplying.*

Problem 3. *Write in your own (casual) words what the word "associativity" means in math for a binary operator.*

Axiom 5 (Commutativity of addition)**.** *For real numbers x, y, we know that $x + y = y + x$.*

Axiom 6 (Commutativity of multiplication)**.** *For real numbers x, y, we know that $x \cdot y = y \cdot x$.*

Problem 4. *Write in your own (casual) words what the word "commutativity" means in math for a binary operator.*

Axiom 7 (Identity of addition)**.** *There is a real number a such that, for all real numbers x, we have $a + x = x + a = x$. We call a the additive identity.*

Axiom 8 (Identity of multiplication)**.** *There is a real number m such that, for all real numbers x, we have $m \cdot x = x \cdot m = x$. We call m the multiplicative identity.*

Problem 5. *Write in your own (casual) words what the word "identity" means in math for a binary operator.*

Problem 6. *Which number is a, the additive identity of $\mathbb{R}$? Which number is m, the multiplicative identity of $\mathbb{R}$?*

Axiom 9 (Inverse of addition)**.** *For a real number x, there is a corresponding real number y such that $y + x = x + y = a$ where a is the additive identity of $\mathbb{R}$. We call y the additive inverse of x.*

Axiom 10 (Inverse of multiplication)**.** *For a non-zero real number x, there is a corresponding real number y such that $y \cdot x = x \cdot y = m$ where m is the multiplicative identity of $\mathbb{R}$. We call y the multiplicative inverse of x.*

Problem 7. *Write in your own (casual) words what the word "inverse" means in math for a binary operator.*

Problem 8. *Given a real number x, what is the additive inverse of x? How does this let us define subtraction?*

Problem 9. *Given a real number x, what is the multiplicative inverse of x? How does this let us define division?*

Axiom 11 (Left distributivity)**.** *We know that multiplication distributes over addition on the left, meaning that $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for real numbers x, y, z.*

Axiom 12 (Right distributivity)**.** *We know that multiplication distributes over addition on the right, meaning that $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ for real numbers x, y, z.*

Problem 10. *What is special about the two distributivity axioms as compared to every other algebraic axiom? Hint: how many operators do the distributivity axioms talk about?*

Phew, that is a lot of axioms. Nonetheless, these algebraic laws of addition and multiplication are hopefully quite natural to you!

However, you might ask: "These axioms are so obvious, what's the point of even writing them down?" Well, we should note that the axioms above do not always apply when restricting from the real numbers to rationals, integers, wholes, or naturals. Also, if we want to create or study other types of "numbers", we may want to check if they satisfy the same algebraic axioms that we are used to.

Problem 11. *Complete the following parts.*

(i) *Is there a natural number a such that $a + x = x + a = x$ for all natural numbers x?*

(ii) *So, does $\mathbb{N}$ have an additive identity a? In other words, is Axiom 7 still true when replacing the word "real" with the word "natural" everywhere?*

(iii) *Among the types of numbers $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, which ones have an additive identity? This is asking if Axiom 7 is still true when replacing the word "real" everywhere with the other types of numbers.*

(iii) *Among the types of numbers $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, which ones have a multiplicative identity? This is asking if Axiom 8 is still true when replacing the word "real" everywhere with the other types of numbers.*

Problem 12. *Complete the following parts.*

(i) *Suppose you have some natural number x. Is there always a natural number, y, that is the additive inverse of x? Why or why not?*

(ii) *So, does $\mathbb{N}$ satisfy the axiom of additive inverses? In other words, is Axiom 9 still true when replacing the word "real" with the word "natural" everywhere?*

(iii) *Among the types of numbers $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, which ones satisfy the axiom of additive inverses? This is asking if Axiom 9 is still true when replacing the word "real" everywhere with the other types of numbers.*

Problem 13. *Complete the following parts.*

(i) *Suppose you have some natural number x. Is there always a natural number, y, that is the multiplicative inverse of x? Why or why not?*

(ii) *So, does $\mathbb{N}$ satisfy the axiom of multiplicative inverses? In other words, is Axiom 10 still true when replacing the word "real" with the word "natural" everywhere?*

(iii) *Among the types of numbers $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, which ones satisfy the axiom of additive inverses? This is asking if Axiom 9 is still true when replacing the word "real" everywhere with the other types of numbers.*

Collecting all of our answers to Problem 11, Problem 12, and Problem 13, we can answer the following problem.

Problem 14. *Fill out the following chart to show which axioms hold when restricting to each type of real number. So, this chart is asking which axioms are true when replacing the word "real" everywhere with other types of numbers. Use a check mark if the axiom holds and an X if it doesn't. Part of the chart is already filled out for you.*

	Naturals	Wholes	Integers	Rationals	Reals
+ closure	✓	✓	✓	✓	✓
+ associativity					✓
+ commutativity					✓
+ identity					✓
+ inverse					✓
· closure	✓	✓	✓	✓	✓
· associativity					✓
· commutativity					✓
· identity					✓
· inverse					✓
Left distributivity					✓
Right distributivity					✓

Remark 1. *The irrational numbers (sometimes denoted by $\mathbb{P}$) are defined to be all of the real numbers that are not rational. Two examples of irrational numbers are π and e. Note that irrational numbers are defined by what they are not, which is why we didn't include them in the diagram on the first page.*

Problem 15 (Extra Credit Challenge!)**.** *Do the same for the irrational numbers. Note that this is much trickier!*

	Irrationals
+ closure	
+ associativity	
+ commutativity	
+ identity	
+ inverse	
· closure	
· associativity	
· commutativity	
· identity	
· inverse	
Left distributivity	
Right distributivity	

Problem 16 (Extra Credit Challenge!)**.** *Which axioms hold for all of the types of numbers $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ but do not hold for the irrationals?*