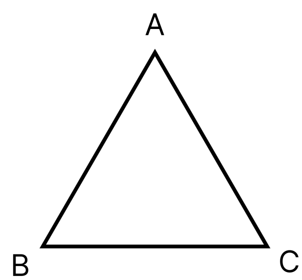
Symmetries and Groups

We've identified the algebraic axioms for the real numbers, and have even mentioned some number systems outside the real line that follow similar axioms. However, there are many other kinds of objects that satisfy these axioms. We first study objects which we can "add" in some way.

Rotation group of an equilateral triangle

Let's say we want to study the symmetries of the equilateral triangle below. To do so, we can start by first only studying the rotations of the triangle about its center.

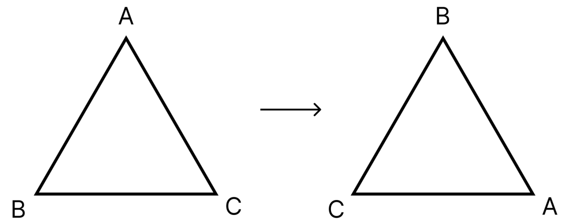


Problem 1. *Suppose I rotate the triangle by 840 degrees counterclockwise. What rotation less than 360 degrees clockwise achieves the same orientation of the triangle?*

Altogether, this lets us study the rotational symmetry of the equilateral triangle by studying the following set.

Definition 1. *The rotation group of this equilateral triangle, written as* R_Δ, *is the set of all clockwise rotations less than 360 degrees about its center that send the vertices back on top of the vertices.*

For example, rotating the triangle by 120 degrees clockwise maps the vertices onto themselves, as below.



Hence, the rotation by 120 degrees clockwise is in the rotation group R_Δ. Let's write rotation by 120 degrees clockwise as r_{120}. Similarly, rotation by 0 degrees clockwise r_0 and rotation by 240 degrees clockwise r_{240} are also in the rotation group R_Δ. It turns out that r_0, r_{60}, r_{120} are the only rotations in the rotation group.

Problem 2. *Why are r_0, r_{60}, r_{120} the only objects in the rotation group?*

We can now define a natural way of adding the objects in the rotation group of the triangle.

Definition 2. *Given two rotations x, y in R_Δ, we define $y+x$ to be the unique rotation z in R_Δ that gives the same orientation of the triangle as given by doing x first and then y.*

Example 1. $r_{120} + r_0 = r_{120}$ *since rotating clockwise by 0 degrees and then rotating clockwise by 120 degrees is the same as just rotating clockwise by 120 degrees.*

Example 2. $r_{120} + r_{240} = r_0$ *since rotating clockwise by 240 degrees and then rotating clockwise by 120 degrees is the same as just not rotating at all.*

Problem 3. *Answer the following equalities.*

$r_0 + r_0 =$ ________ $r_0 + r_{120} =$ ________ $r_0 + r_{240} =$ ________
$r_{120} + r_0 =$ ________ $r_{120} + r_{120} =$ ________ $r_{120} + r_{240} =$ ________
$r_{240} + r_0 =$ ________ $r_{240} + r_{120} =$ ________ $r_{240} + r_{240} =$ ________

From Problem 3, we conclude that R_Δ with its definition of addition satisfies the following axioms.

Axiom 1 (Closure of addition)**.** *For rotations x, y in R_Δ, we know that $x+y$ is also in R_Δ.*

Axiom 2 (Commutativity of addition)**.** *For rotations x, y in R_Δ, we know that $x + y = y + x$.*

Axiom 3 (Identity of addition)**.** *There is a rotation a in R_Δ such that, for any rotation x in R_Δ, we have $a + x = x + a = x$. We call a the additive identity.*

Problem 4. *Which number is a, the additive identity of R_Δ?*

Axiom 4 (Inverse of addition)**.** *For any rotation x in R_Δ, there is a corresponding rotation $-x$ in R_Δ such that $(-x) + x = x + (-x) = a$ where a is the additive identity of R_Δ. We call $-x$ the additive inverse of x.*

Problem 5. *Answer the following equalities.*

$$-r_0 = \underline{\qquad\qquad} \qquad -r_{120} = \underline{\qquad\qquad} \qquad -r_{240} = \underline{\qquad\qquad}$$

If you check a little bit more, you can prove the following fact as well (which is obvious geometrically).

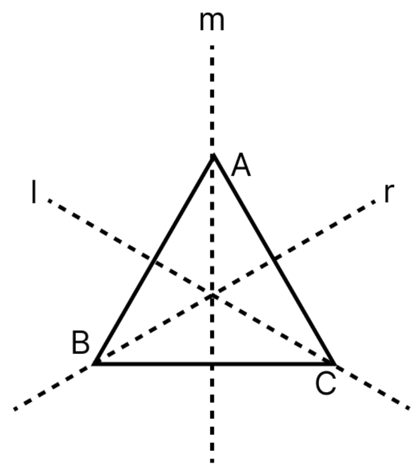
Axiom 5 (Associativity of addition)**.** *For rotations x, y, z in R_Δ, we know that $x + (y + z) = (x + y) + z$. This lets us ignore parentheses when adding.*

Altogether, R_Δ is a set with a way of adding that satisfies all of the same addition axioms that the real numbers do!

Symmetry group of an equilateral triangle

We originally wanted to study all of the Euclidean symmetries of the equilateral triangle. Well, these must be combinations of rotations (about the center) and reflections (across axes passing through the triangle's center) that map vertices onto vertices. Note that only combinations of rotations and reflections are possible since these are the only transformations that preserve distances and angles (excluding translations, which do not map vertices onto vertices).

With a bit of geometric intuition, one can see that the only reflections of the triangle that map vertices onto vertices are f_l, f_m, f_r which are, respectively, the reflections across the lines l, m, r in the diagram below.



This observation lets us define the following.

Definition 3. *Let F_Δ denote the set of f_l, f_m, f_r. The symmetry group of the equilateral triangle, written S_Δ, is the set of symmetries in R_Δ and the symmetries in F_Δ. Given two symmetries x, y in S_Δ, we define $y + x$ to be the unique symmetry z in S_Δ that gives the same orientation of the triangle as given by doing x first and then y.*

This familiar definition of addition can be justified (meaning, in particular, that addition in S_Δ satisfies the closure axiom) by our calculations in Problem 3 and our calculations in the following problem.

Problem 6. *Answer the following equalities.*

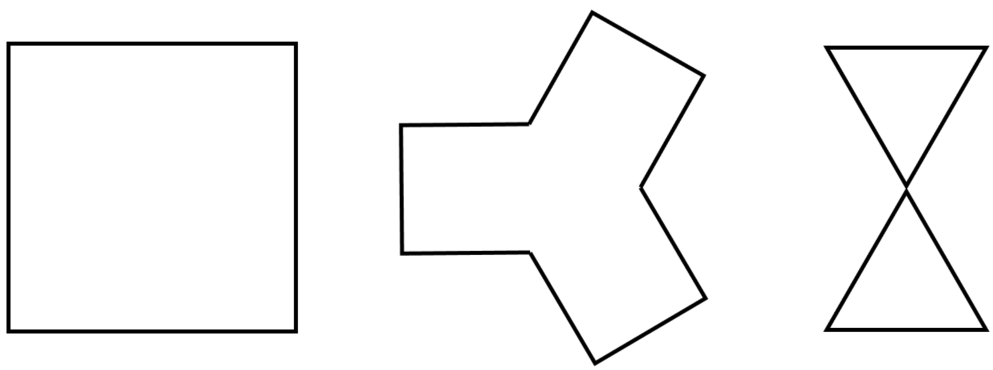
$f_l + f_m =$ ________	$f_l + f_r =$ ________	$f_m + f_l =$ ________
$f_m + f_r =$ ________	$f_r + f_l =$ ________	$f_r + f_m =$ ________
$r_0 + f_l =$ ________	$r_0 + f_m =$ ________	$r_0 + f_r =$ ________
$f_l + r_0 =$ ________	$f_m + r_0 =$ ________	$f_r + r_0 =$ ________
$r_{120} + f_l =$ ________	$r_{120} + f_m =$ ________	$r_{120} + f_r =$ ________
$f_l + r_{120} =$ ________	$f_m + r_{120} =$ ________	$f_r + r_{120} =$ ________
$r_{240} + f_l =$ ________	$r_{240} + f_m =$ ________	$r_{240} + f_r =$ ________
$f_l + r_{240} =$ ________	$f_m + r_{240} =$ ________	$f_r + r_{240} =$ ________

Again, geometric intuition (supported by many tedious computations) does show that addition in S_Δ satisfies associativity. In turn, Problem 3 along with Problem 6 imply that addition in S_Δ satisfies four out of all five addition axioms.

Problem 7. *Which addition axiom is not satisfied in S_Δ?*

Symmetric groups of other shapes

In class, the instructors will pass out cutouts of the following three shapes.



You can define the symmetry groups of each of these shapes in the same way as we did with the equilateral triangle: the union of the vertex-preserving rotations and reflections. You can also define addition in the symmetry groups in the same way. After playing around, you will see that addition in each of these symmetry groups also satisfies the closure, associativity, identity, and inverse axioms.

Problem 8. *Exactly one symmetry group for the three shapes above has addition which satisfies the commutativity axiom. Which is it?*

Play around with the cutouts for the remainder of class to learn more about these symmetry groups! The instructors will test your understanding along the way.