Basic Combinatorics and Probability

Kason Ancelin and Sanjit Dandapanthula

January 15, 2022

Ice-breaker.

Today, the ice-breaker will be quite short. Simply guess the number of possible Blaze "build your own" pizzas where you choose from a selection of crusts, sauces, cheeses, toppings, and finishes. The exact parameters will be given later, but just guess for right now. Now, also guess the number of possible 9 by 9 Sudoku boards.

Introduction.

Today, we will provide an in-depth introduction to the basics of combinatorics, a field of mathematics all about counting! We begin with a motivating example. Let's say you are robbing a bank and you have passed by all the tellers. The only thing standing between you and a multi-million dollar payout is a safe with a 4-digit combination. You have a henchman named Cornelius who can punch in 50 unique codes a minute. Furthermore, being the smart criminal you are, you selected a bank in the middle of nowhere that will take the police 3 hours and 30 minutes to arrive at and intervene. The question is, can you be certain that Cornelius will open the safe before the police arrive? We will build some basic principles and background and then come back and solve this problem.

Let's see another basic problem to build some theory. If you have two pairs of shorts, one blue and one black, and three T-shirts, red, white and yellow, how many outfits can you make? Let's see how many different combinations of T-shirts and shorts we can have.

1. Blue shorts and ________ T-shirt

2. Blue shorts and ________ T-shirt

3. Blue shorts and ________ T-shirt

4. Black shorts and ________ T-shirt

5. Black shorts and ________ T-shirt

6. Black shorts and ________ T-shirt

This is 6 outfits, or 6 combinations, in all. Sometimes it can become very tedious to list out all possible combinations, as the following example illustrates.

If there are four chairs and four students (say Ashley, Benjamin, Caitlyn and Daniel) in a classroom, in how many ways can the students be seated on the chairs?

1. How many different students have the option of sitting on the first chair? ________

2. If one of them sits on the first chair, how many can sit on the second? ________

3. Now, how many can sit on the third chair? ________

4. How many can sit on the fourth chair? ________

So, let's see how many total ways of arranging the students on the chairs there are. For each choice of the student who sits on the first chair (4 in total), we can then consider all the choices of who can sit on the second (3 in total). This gives us 4×3 different arrangements for the first two chairs. Then, for all the possible choices of who can sit on the first and the second chairs, we can consider all the choices of who can sit on the third chair (2 in total). So, we can have $4 \times 3 \times 2$ arrangements for the first three chairs. Then, for the different possible arrangements of students sitting on the first three chairs, we can consider how many students can sit on the last chair. Well, since all the other chairs are taken and there is only one student left, there is only one way to place this final, fourth student This gives us ________ total arrangements for the four chairs ($4 \times 3 \times 2 \times 1$ is sometimes written as $4!$). So, these are ways in which all the students can be seated on the four chairs.

You may now recognize, that in the first example, we could use the same idea as example 2 displayed. There are two colors of shorts to choose from and 3 colors of shirts to choose and thus the number of shorts-shirts combinations is simply $3 * 2 = 6$. This idea is known as the Multiplication Principle of Combinatorics. Stated simply, if there are x ways of doing something and y ways of doing another thing, then there are xy ways of performing both actions. Returning to our motivating example, we will solve our dilemma by using the Multiplication Principle! For the first digit, there are 10 options that can be selected (0-9 inclusive). For the second, there are also 10. We quickly observe that for each digit there are 10 options. So there are a total of 10^4 possible combinations to the safe. We'll leave the rest of the arithmetic of the problem to you to fully solve it! After solving the original problem, solve it for if the code was only 3 digits long? How about an 8 digit code? There are 10^4 total codes so it will take at most

200 minutes to find the right code, beating the 210 minutes the police will take. For 3 digits, it will take 20 minutes, and for 8 digits, it will take 2,000,000 minutes which is roughly 3.8 years.

Exercises. The following exercises primarily use the multiplication principle.

Exercise 1. *A sweater has to be knitted with three different colors of yarn: one for the left sleeve, one for the right sleeve and one for the body. If we have nine differently colored yarns available, how many different kinds of sweaters can be made?*

$9 * 8 * 7$

Exercise 2. *Blaze carries 5 kinds of crusts, 5 different sauces, 8 cheeses, 26 toppings, and 6 finishes for after the pizza has cooked. How many different "simple" pizzas are sold at Blaze (a simple pizza is one that has 1 crust, 1 sauce, 1 cheese, 1 topping, and 1 finish)? We will return to this at the end to find* *the total number of pizzas Blaze sells! To add excitement to solving this problem, after solving the simple pizza problem, alter your prediction for how many total "byo" pizzas are possible at Blaze? A "byo" pizza is defined as an item with 1 crust, 1 sauce or no sauce, any number of cheeses, any number of toppings, and any number of finishes.*

$5 * 5 * 8 * 26 * 6$

Exercise 3. *There are three ways of going from New York to Los Angeles: by train, by plane and by teleportation. And there are three ways of going from Los Angeles to Hawaii: by plane, by teleportation and by sea. How many ways are there to go from New York to Hawaii via Los Angeles?*

9

Exercise 4. *Each box in a 3×3 square can be colored using one out of three colors. How many different colorings of the grid are there?*

3^9

We now present another intuitive principle to help us count!

Max wants to go from New York to Hawaii. He has two options. One, he can go from New York to LA and then to Hawaii, like we did above. Or, he can go from New York to Chicago, where his grandmother lives, and then to Hawaii. There are four ways to get from New York to Chicago: by train, by car, by teleportation or by plane. And there are two ways to get from Chicago to Hawaii: by plane and by teleportation. In how many ways can Max get to Hawaii from New York? All we need to do is answer the following 2 questions and add them!

How many ways are there to go from New York to Hawaii via Chicago? 8

How many ways are there to go from New York to Hawaii via LA? 9

And so we can now answer: How many ways are there to go from New York to Hawaii in all? 17

This is the Addition Principle of Combinatorics. Let's now practice using this real quick!

Exercise 5. *SJ has to draw one card from a deck with 52 playing cards. In how many ways can he choose a heart or diamond card?*

26

Exercise 6. *A college acting troupe has 6 junior women, 8 junior men, 5 senior women and 4 senior men. In how many ways can the teacher select a senior couple or a junior couple to play the roles of Romeo and Juliet?*

68

Exercise 7. *In a movie rental store, there are five different English movies, three French movies, two Russian movies and four Hindi movies. How many ways are there to rent 3 movies in 3 different languages?*

$5 * 3 * 2 + 5 * 3 * 4 + 5 * 2 * 4 + 3 * 2 * 4$

Permutations vs. Combinations

Sometimes when we count, we care about the order of the elements selected, and sometimes we don't. For instance, if we are selecting a two-person team from players A, B, C, and D, selecting A then B generates the same team as selecting B then A. Here, if we were counting the total number of teams that could be made, we see that the order of team member selection does not matter. However, in counting the total number of possible 4-digit iPhone passcodes, order must certainly matters! If your passcode is 7189, 8917 will not let you into your phone despite using the same digits. Since these codes are not considered the same, we observe that the order of the digits matters. *Permutations* are an arrangement of elements in order, so when counting the number of permutations **ORDER MATTERS**. *Combinations* have no order associated with them, and so order does not matter.

Exercise 8. *How many permutations are there of a set of n distinct elements? Hint: Use factorials. k! = k * (k-1) * (k-2) * ... 1. k! is read as k-factorial.*

$n!$

Exercise 9. *You are on the Price Is Right and the host gives you the following game: Take these 5 items, and place them in order from least expensive to most expensive (each item has a different price). (a) How many different ways could you place the items in this game? Now assume you know that item A is less expensive than item B, but nothing else. (b) Now how many orderings would you reasonably play? Finally, assume again that you know item A is less expensive than item B but you also know item A is in position 2 (2nd cheapest). (c) Now how many orderings?*

$5!$

$4! + 3 * 3! + 2 * 3 * 2! + 3!$

$3 * 3!$

Exercise 10. *Finally, consider the previous Price Is Right problem but where you know A and B are exactly the same price and all other items have distinct prices. Now how many orderings could you reasonably play?*

$2 * 4!$ (2 is from every order having either A then B right next to each other or B then A right next to each other)

Binomial Coefficients

Suppose you are asked how many ways are there to choose a team of 5 people from a group of 9 people? One approach is to start counting all the possibilities, however, I would warn you against it since there are 126 ways! A better approach may be to think about using permutations. Let's begin by counting the number of permutations of all 9 people (i.e. listing them in some order). From the previous exercise above, we know there are 9! ways to do this. Now, let's just say that we generate a team by taking the first 5 members in the list for our team. So far, each list will generate some team, however, simply counting the number of permutations of all 10 individuals severely overcounts our desired number of teams since, for example, all of the following permutations generate the same team:

ABCDEFGHI ABCDEFGHI AEBCDHGFI BEDACFIHG EBADCIFGH

Above displays just some of these permutations generating team $ABCDE$. Now the question becomes, can we find the number of ways that just counting the number of permutations of all 9 individuals re-counts a given team. If we can do so, then we will be able to solve our desired problem. So, for a given team, say $ABCDE$, how many different permutations of the 9 individuals generate this team? Well, as long as A, B, C, D, and E are in the first 5 positions and F, G, H, and I are in the last 4 positions the permutation will generate this team. So, by the multiplication principle, there are $5! * 4!$ different permutations to generate this team, and so each team is counted a total of $5! * 4!$ times. Thus, the total number of possible teams is just $\frac{9!}{5! \cdot 4!}$. More generally, if we were choosing k unordered elements from n total elements (where $k \leq n$), the number of ways to do this is $\frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$. This expression is usually written as *binomnk* (read n choose k) and is known as a binomial coefficient. This is super important!

Exercise 11. *Run through the logic above with finding the number of ways of selecting the 5 person team above but instead of using the numbers 5 and 9 use k and n and justify the equation given above to yourself.*

Exercise 12. *There are 10 people in a room, and each of them shakes everyone else's hand. How many total handshakes occurred?*

$\binom{10}{2}$

Exercise 13. *How many 5 card poker hands are possible from a standard 52-card deck?*

$\binom{52}{5}$

Exercise 14. *You are enrolling in classes for school. There are 15 total classes you can take and you have to take exactly 7 classes. How many combinations of classes can you take? Now, how many permutations of classes can you take (hint: use binomial coefficient and then multiply it by something, think through how we constructed the binomial coefficient to figure out what this"thing" is)? How about both these questions if you could choose to either take 6 or 7 classes?*

$\binom{15}{7}$

$\frac{15!}{8!}$

$\binom{15}{7} + \binom{15}{6}$

$\frac{15!}{8!} + \frac{15!}{9!}$

Note: Something valuable came here, to find the number of permutations of k elements from a group of n total elements we simply need to calculate $\frac{n!}{(n-k)!}$ or equivalently $\binom{n}{k} * k!$. Don't forget this! It is quite useful!

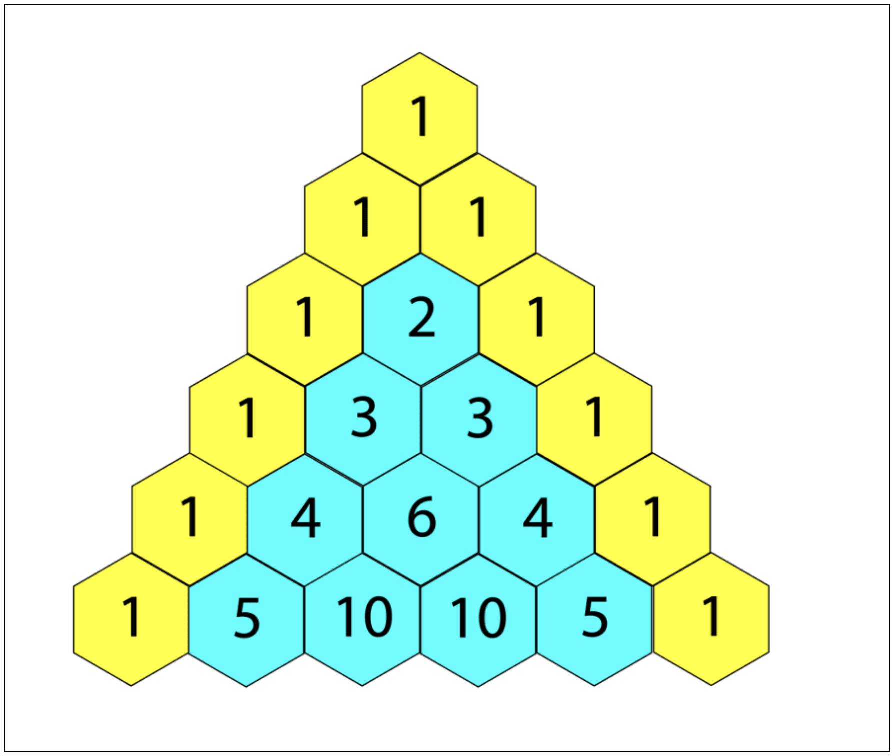
Exercise 15. *For $(x+y)^{10}$, what is the coefficient of the term x^4y^6. How about x^6y^4? Explain how these values relate. Find the coefficient for x^ky^{10-k} and $x^{10-k}y^k$ for each k?*

$\binom{10}{4}$ for both

$\binom{n}{k} = \binom{n}{n-k}$ is what should be realized

Exercise 16. *Complete Pascal's Triangle below (add 2-3 more rows). Pascal's Triangle is the coefficient on the terms of $(x+y)^n$ in order as so: $x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + y^n$ in the nth row. So the first represents $(x+y)^0 = 1$, then $(x+y)^1 = 1x+1y$, then $(x+y)^2 = 1x^2+2xy+1y^2$, then $(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$, and so on. We have started it below for you, and want you to play around with it and attempt to find some patterns in it. Take 5 minutes and explore.*

Let's end up showing them the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ through a counting argument, and then ask them to try and do the arithmetic. Also, have them notice the symmetry of the tree, and the nth row sums to 2^n. Maybe take a quick 5 minutes in class and show them this stuff on the board after they have attempted to find it themselves.



What do we notice about this triangle? Come up with an identity relating binomial coefficients. Notice anything about the sums of the rows? Any symmetries?

Now it is time to return to the pizza problem. Calculate the number of possible build-your-own pizzas at Blaze. A "byo" pizza is defined as an item with 1 crust, 1 sauce or no sauce, any number of cheeses, any number of toppings, and any number of finishes (refer back to page 3 for the number of each ingredient).

After calculating this, come up with some number to put this value into perspective. For instance, one could be: there are more possible Blaze pizzas than the United States GDP in a year in dollars.

$5 * 6 * 2^8 * 2^{26} * 2^6 = 32,985,348,833,280$

Challenging counting problems:

Exercise 17. *Find the coefficient of x^7y^2 in the expansion of $(2y - x)^9$.*

$2^2 * (-1)^7 * \binom{9}{2}$

Exercise 18. *There are 10 chairs setup in a circle and you and 9 other of your friends want to sit in these chairs. How many ways can you sit in these chairs if two arrangements are considered the same if they can be rotated onto each other, reversed order (i.e. counterclockwise instead of clockwise), or a combination of both. For instance, $ABCDEFGHIJ$, $BCDEFGHIJA$, $FGHIJABCDE$, and $EDCBAJIHGF$ are all considered the same arrangement. Intuitively, you can think of this as if there is some way to read these letters in the same order, they are the same arrangement.*

$\frac{(n-1)!}{2}$

Exercise 19. *A toy consists of a ring with 3 red beads and 7 blue beads on it. If two configurations of beads differ only by rotations and reflections, they are considered the same toy (just as in the above problem). How many different toys are there?*

To solve this problem in full generality, I believe you need to use Burnside's lemma. This version of the problem will be very difficult for students. If students ask about this problem, you can give them an easier case to solve (like with 2 red beads and 3 blue beads).

Exercise 20. *You have 15 slices of pizza and 4 friends sharing it. How many different ways can the 4 friends entirely finish the pizza? For example, friend A could eat 15 slices and everyone else eats none or friend A eats 3, B eats 4, C eats 3, and D eats 5, and so on.*

Stars and bars, $\binom{18}{3}$

Exercise 21. *Now assume that every person gets at least one slice. Now how many ways are there?*

$\binom{14}{3}$

Using Combinatorics to Solve Probability Problems

We will now use what we learned in the previous sections to solve probability problems. In many of these problems, you will want to count the number of successful outcomes and divide them by the number of total possible outcomes, when all outcomes are equally likely. For instance, when flipping a fair coin, rolling a fair die, and uniformly choosing an item.

$$P[event] = \frac{\text{number of success}}{\text{total outcomes}}$$

Exercise 22. *You flip a fair coin 4 times. What is the probability of flipping exactly two heads and two tails?*

$\frac{3}{8}$

Exercise 23. *You flip a fair coin 10 times. What is the probability of flipping 4, 5, or 6 heads?*

$\frac{672}{1024}$ simplify it if necessary, but this is what the counting will yield.

Exercise 24. *There are 10 people randomly standing in a line. What is the probability they are standing in age order (either oldest to youngest or youngest to oldest)?*

$\frac{2}{10!}$

Exercise 25. *A particle starts at the origin and then at each step either takes one step to the left with probability $\frac{1}{2}$ or one step to the right with probability $\frac{1}{2}$. After two steps, what is the probability that the particle is back at origin? How about after 4 steps? 6 steps? How about the probability of being at the origin after an odd number of steps?*

$\frac{1}{2}$ $\frac{3}{8}$ $\frac{5}{16}$

Never at origin after an odd number of steps.

Exercise 26. *Your friend rolls 3 dice and gets a sum of 13. What is the probability that the sum of your 3 dice rolls (roll 1 die 3 times, one after the other, and add up their values) will be greater than your friends?*

$15 + 10 + 6 + 3 + 1 = 35$

Challenge Problems.

Exercise 27. *You flip an unfair coin (comes up head with probability p and tails with probability 1-p) 4 times. What is the probability of flipping exactly two heads and two tails? (hint: count the number of ways to arrange two heads and two tails and then find the probability of one of these outcomes occurring).*

$\binom{4}{2} * p^2 * (1-p)^2$

Exercise 28. *In the game, Risk, an attacker rolls 3 dice and the defender rolls 2 dice in a battle (all dice rolled together at the same time). The attacker and defender then line up their dice in order. If the largest face up defending die is greater than or equal to the largest attacking face up die, the defender kills an attacker. Otherwise, the attacker kills a defender. Similarly, if the 2nd largest face up defending die is greater than or equal to the 2nd largest attacking face up die, the defender kills an attacker. Otherwise, the attacker kills a defender. What is the probability that the attacker kills two defenders in a battle? What is the probability that the defender kills two attackers in a battle? What is the probability that they each kill one of each other?*

The solution is very involved and messy. I will come with a piece of paper solving it to math circle but unfortunately do not have the time to typeset it here. It isn't horribly difficult, just a bit messy.

Exercise 29. *The National Weather Service predicts a person has a 1-in-15300 chance of getting struck by lightning in life. Guess how many 6's would need to be rolled in a row until the probability of getting such a streak was less likely than getting struck by lightning in a lifetime. Now calculate it.*

6 rolls

Exercise 30. *Amy has a number of rocks such that the mass of each rock, in kilograms, is a positive integer. The sum of the masses of the rocks is 2018 kilograms. Amy realizes that it is impossible to divide the rocks into two piles of 1009 kilograms. What is the maximum possible number of rocks that Amy could have?*

https://www.cantorsparadise.com/an-olympiad-combinatorics-problem-with-a-beautiful-geometric-solution-caf52dbfcde1

This is for just in case someone completes the whole packet, this should be pretty difficult and stall them for a while. The link above describes the solution.