ORMC Olympiad Group
Winter: Week 2
Analysis: Functions

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Problems

1. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x+y) = f(x) + f(y)$.

2. (AMC12 2011A) Let $f_1(x) = \sqrt{1-x}$, and for integers $n \geq 2$, let $f_n(x) = f_{n-1}(\sqrt{n^2 - x})$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $[c]$. What is $N + c$?

   (A) -226 **(B)** -144 **(C)** -20 **(D)** 20 **(E)** 144

3. **(PSS Functions P4)** Prove that function f is periodic if for some $a > 0$ and all $x \in \mathbb{R}$
$$f(x+a) = \frac{1+f(x)}{1-f(x)}$$

4. (AHSME 1975) Suppose $f(x)$ is defined for all real numbers x; $f(x) > 0$ for all x, and $f(a)f(b) = f(a+b)$ for all a and b. Which of the following statements is true?

   I. $f(0) = 1$ II. $f(-a) = 1/f(a)$ for all a III. $f(a) = \sqrt[3]{f(3a)}$ for all a IV. $f(b) > f(a)$ if $b > a$

   (A) III and IV only **(B)** I, III, and IV only **(C)** I, II, and IV only **(D)** I, II, and III only **(E)** All are true.

5. **(IMOmath P3)** Function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x + f(x)$ for all $x \in \mathbb{R}$. Find all solutions of the equation $f(f(x)) = 0$

6. **(IMOmath P1)** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x + y) + 1$.

7. **(Turkish JNMO 2000 Second Round P3)** $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation
$$f(x)f(y) - af(xy) = x + y$$
for every real numbers x, y. Find all possible real values of a.

8. **(AHSME 1975-modified)** Suppose $f(x)$ is a function defined for all real numbers x; so that $f(x) > 0$ for all x, and $f(a)f(b) = f(a + b)$ for all a and b. Among all such functions, determine if the following statement is always true.
$$f(b) \geq f(a) \text{ for all } b > a$$

9. **(Turkish NMO 2004 Second Round P4)** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$

10. **(IMO 1977)** Let $f(n)$ be a function defined on the set of all positive integers and with all its values in the same set. Prove that if
$$f(n + 1) > f(f(n))$$
for each positive integer n, then $f(n) = n$ for each n.

11. **(Turkish NMO 2008 Second Round P4)** $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ satisfy the given conditions

    $a)$ $f(0, 0) = 1$, $f(0, 1) = 1$,

    $b)$ $\forall k \notin \{0, 1\}$ $f(0, k) = 0$ and

    $c)$ $\forall n \geq 1$ and k , $f(n, k) = f(n - 1, k) + f(n - 1, k - 2n)$

    find the sum $\displaystyle\sum_{k=0}^{\binom{2009}{2}} f(2008, k)$