ORMC AMC 10/12 Group
(More) Circles

January 15, 2023

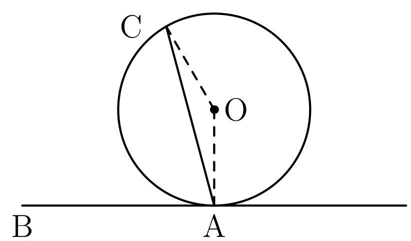
1 Warmup: Review

1. **(2020 AMC 12B # 10)** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

   Note: Last week we did this using the Inscribed Angle Theorem– this time, use Power of a Point.

   Let N be the intersection of AB with ω. Then, by power of a point, we have $AN^2 = AP \cdot AM$, which means that $1/4 = AP \cdot (\sqrt{5}/2)$. So, $AP = 1/2\sqrt{5} = \boxed{\sqrt{5}/10}$

2. Find the measure of $\angle BAC$ in terms of θ, where $\theta = m\overset{\frown}{AC}$:



   Draw the auxiliary lines AO and OC. Note that $m\angle CAO = m\angle ACO = 90° - \frac{\theta}{2}$. Since $\angle CAO$ and $\angle BAC$ are clearly complementary, we have $\boxed{m\angle BAC = \dfrac{\theta}{2}}$.

3. In unit square $ABCD$, point P is chosen at random. What is the probability that $\triangle APB$ is an obtuse triangle?

   By the inscribed angle theorem, any right triangle would have P on the semicircle with diameter AB. So, clearly, any triangle with P inside this semicircle is obtuse, and any triangle with P outside is acute. So, the probability that $\triangle ABP$ is obtuse is the area of the semicircle, $\frac{\pi}{2}$, divided by the area of the square, 1. Thus, the probability is $\boxed{\dfrac{\pi}{2}}$.

4. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.

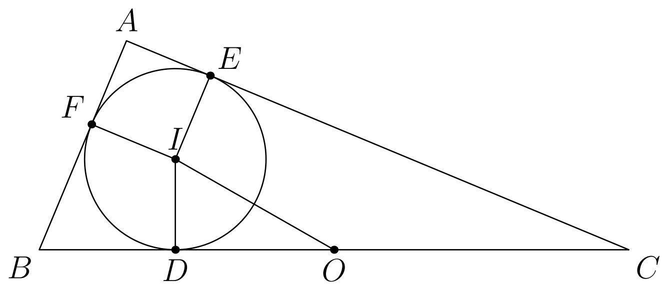
   Clearly, we want to use power of a point. Let N be the midpoint of BC. Then, we have $AB \cdot x = CN^2 \implies 5\sqrt{5}x = 5^2 \implies x = \boxed{\sqrt{5}}$.

   (Alternatively, note that this is the same problem as (1), but scaled up by a factor of 10.)

2 Triangle Incircles and Circumcircles

2.1 Examples

1. **(2004 AMC 10B #22)** A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?



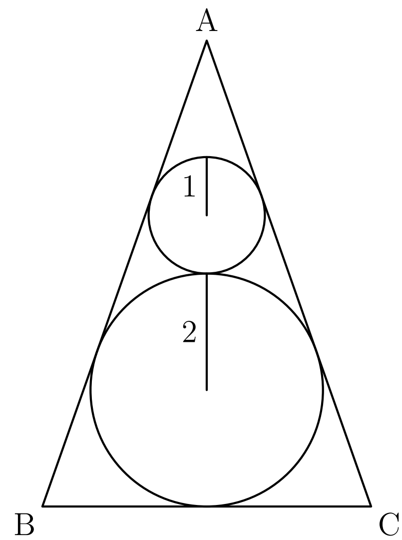
This is a right triangle. Pick a coordinate system so that the right angle is at $(0, 0)$ and the other two vertices are at $(12, 0)$ and $(0, 5)$.

As this is a right triangle, the center of the circumcircle is in the middle of the hypotenuse, at $(6, 2.5)$.

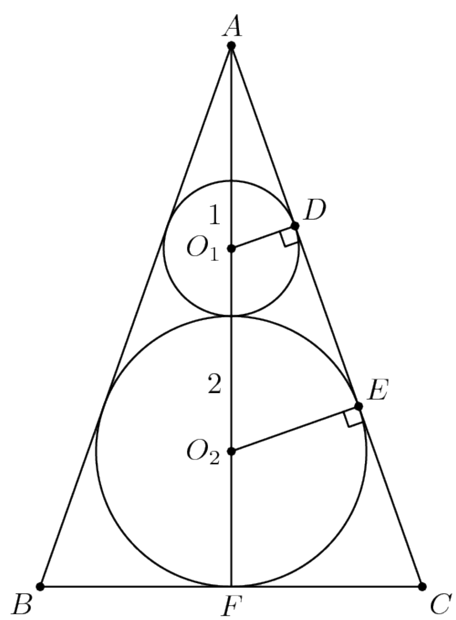
The radius r of the inscribed circle can be computed using the well-known identity $\frac{rP}{2} = S$, where S is the area of the triangle and P its perimeter. In our case, $S = \frac{5 \cdot 12}{2} = 30$ and $P = 5 + 12 + 13 = 30$. Thus, $r = 2$. As the inscribed circle touches both legs, its center must be at $(r, r) = (2, 2)$.

The distance of these two points is then $\sqrt{(6-2)^2 + (2.5-2)^2} = \sqrt{16.25} = \sqrt{\frac{65}{4}} = \boxed{\frac{\sqrt{65}}{2}}$.

2. **(2006 AMC 10A #16)** A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



Let the centers of the smaller and larger circles be O_1 and O_2 , respectively. Let their tangent points to $\triangle ABC$ be D and E, respectively. We can then draw the following diagram:



We see that $\triangle ADO_1 \sim \triangle AEO_2 \sim \triangle AFC$. Using the first pair of similar triangles, we write the proportion:

$\frac{AO_1}{AO_2} = \frac{DO_1}{EO_2} \implies \frac{AO_1}{AO_1+3} = \frac{1}{2} \implies AO_1 = 3$ By the Pythagorean Theorem, we have $AD = \sqrt{3^2 - 1^2} = \sqrt{8}$.

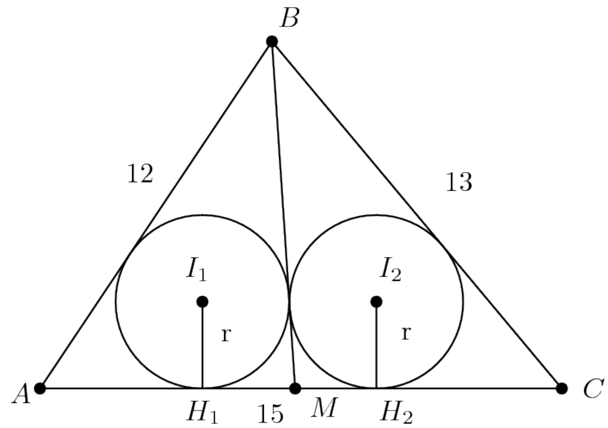
Now using $\triangle ADO_1 \sim \triangle AFC$,

$\frac{AD}{AF} = \frac{DO_1}{FC} \implies \frac{2\sqrt{2}}{8} = \frac{1}{FC} \implies FC = 2\sqrt{2}$ Hence, the area of the triangle is

$$\frac{1}{2} \cdot AF \cdot BC = \frac{1}{2} \cdot AF \cdot (2 \cdot CF) = AF \cdot CF = 8\left(2\sqrt{2}\right) = \boxed{\textbf{(D) } 16\sqrt{2}}$$

2.2 Exercises

1. **(2010 AIME I #15)** In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Then $\frac{AM}{CM} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



Let $AM = x$, then $CM = 15 - x$. Also let $BM = d$ Clearly, $\frac{[ABM]}{[CBM]} = \frac{x}{15-x}$. We can also express each area by the rs formula. Then $\frac{[ABM]}{[CBM]} = \frac{p(ABM)}{p(CBM)} = \frac{12+d+x}{28+d-x}$. Equating and cross-multiplying yields $25x + 2dx = 15d + 180$ or $d = \frac{25x-180}{15-2x}$. Note that for d to be positive, we must have $7.2 < x < 7.5$.

By Stewart's Theorem, we have $12^2(15-x) + 13^2x = d^2 15 + 15x(15-x)$ or $432 = 3d^2 + 40x - 3x^2$. Brute forcing by plugging in our previous result for d, we have $432 = \frac{3(25x-180)^2}{(15-2x)^2} + 40x - 3x^2$. Clearing the fraction and gathering like terms, we get $0 = 12x^4 - 340x^3 + 2928x^2 - 7920x$.

Aside: Since x must be rational in order for our answer to be in the desired form, we can use the Rational Root Theorem to reveal that $12x$ is an integer. The only such x in the above-stated range is $\frac{22}{3}$.

Legitimately solving that quartic, note that $x = 0$ and $x = 15$ should clearly be solutions, corresponding to the sides of the triangle and thus degenerate cevians. Factoring those out, we get $0 = 4x(x-15)(3x^2 - 40x + 132) = x(x-15)(x-6)(3x-22)$. The only solution in the desired range is thus $\frac{22}{3}$. Then $CM = \frac{23}{3}$, and our desired ratio $\frac{AM}{CM} = \frac{22}{23}$, giving us an answer of $\boxed{045}$.

2. **(2011 AIME II #13)** Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120^\circ$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Denote the midpoint of $\overline{DC}$ be E and the midpoint of $\overline{AB}$ be F. Because they are the circumcenters, both Os lie on the perpendicular bisectors of AB and CD and these bisectors go through E and F.

It is given that $\angle O_1PO_2 = 120^\circ$. Because O_1P and O_1B are radii of the same circle, the have the same length. This is also true of O_2P and O_2D. Because $m\angle CAB = m\angle ACD = 45^\circ$, $m\,\overset{\frown}{PD} = m\,\overset{\frown}{PB} = 2(45^\circ) = 90^\circ$. Thus, O_1PB and O_2PD are isosceles right triangles. Using the given information above and symmetry, $m\angle DPB = 120^\circ$. Because ABP and ADP share one side, have one side with the same length, and one equal angle, they are congruent by SAS. This is also true for triangle CPB and CPD. Because angles APB and APD are equal and they sum to 120 degrees, they are each 60 degrees. Likewise, both angles CPB and CPD have measures of 120 degrees.

Because the interior angles of a triangle add to 180 degrees, angle ABP has measure 75 degrees and angle PDC has measure 15 degrees. Subtracting, it is found that both angles O_1BF and O_2DE have measures of 30 degrees. Thus, both triangles O_1BF and O_2DE are 30-60-90 right triangles. Because F and E are the midpoints of AB and CD respectively, both FB and DE have lengths of 6. Thus, $DO_2 = BO_1 = 4\sqrt{3}$. Because of 45-45-90 right triangles, $PB = PD = 4\sqrt{6}$.

Now, letting $x = AP$ and using Law of Cosines on $\triangle ABP$, we have

$$96 = 144 + x^2 - 24x\frac{\sqrt{2}}{2}$$
$$0 = x^2 - 12x\sqrt{2} + 48$$

Using the quadratic formula, we arrive at

$$x = \sqrt{72} \pm \sqrt{24}$$

Taking the positive root, $AP = \sqrt{72} + \sqrt{24}$ and the answer is thus $\boxed{096.}$

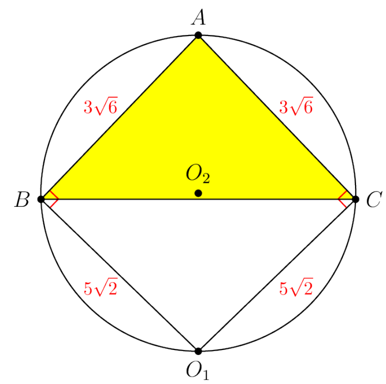
3 Cyclic Quadrilaterals

3.1 Examples

1. **(2021 Fall AMC 10A #15)** Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

   Let $\odot O_1$ be the circle with radius $5\sqrt{2}$ that is tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. Note that $\angle ABO_1 = \angle ACO_1 = 90^\circ$. Since the opposite angles of quadrilateral ABO_1C are supplementary, quadrilateral ABO_1C is cyclic.

   Let $\odot O_2$ be the circumcircle of quadrilateral ABO_1C. It follows that $\odot O_2$ is also the circumcircle of $\triangle ABC$, as shown below: By the Inscribed Angle Theorem, we conclude that $\overline{AO_1}$ is the

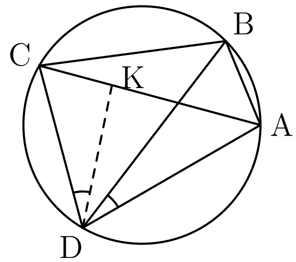


   diameter of $\odot O_2$. By the Pythagorean Theorem on right $\triangle ABO_1$, we have

$$AO_1 = \sqrt{AB^2 + BO_1^2} = 2\sqrt{26}.$$

   Therefore, the area of $\odot O_2$ is $\pi \cdot \left(\frac{AO_1}{2}\right)^2 = \boxed{\textbf{(C) } 26\pi}$.

2. **(2015 AMC 12B #19)** In $\triangle ABC$, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $CBWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?



In order to solve this problem, we can search for similar triangles. Begin by drawing triangle ABC and squares $ABXY$ and $ACWZ$. Draw segments $\overline{YZ}$ and $\overline{WX}$. Because we are given points X, Y, Z, and W lie on a circle, we can conclude that $WXYZ$ forms a cyclic quadrilateral. Take $\overline{AC}$ and extend it through a point P on $\overline{YZ}$. Now, we must do some angle chasing to prove that $\triangle WBX$ is similar to $\triangle YAZ$.

Let α denote the measure of $\angle ABC$. Following this, $\angle BAC$ measures $90-\alpha$. By our construction, $\overline{CAP}$ is a straight line, and we know $\angle YAB$ is a right angle. Therefore, $\angle PAY$ measures α. Also, $\angle CAZ$ is a right angle and thus, $\angle ZAP$ is a right angle. Sum $\angle ZAP$ and $\angle PAY$ to find $\angle ZAY$, which measures $90+\alpha$. We also know that $\angle WBY$ measures $90+\alpha$. Therefore, $\angle ZAY = \angle WBX$.

Let β denote the measure of $\angle AZY$. It follows that $\angle WZY$ measures $90+\beta°$. Because $WXYZ$ is a cyclic quadrilateral, $\angle WZY + \angle YXW = 180°$. Therefore, $\angle YXW$ must measure $90-\beta$, and $\angle BXW$ must measure β. Therefore, $\angle AZY = \angle BXW$.

$\angle ZAY = \angle WBX$ and $\angle AZY = \angle BXW$, so $\triangle AZY \sim \triangle BXW$! Let $x = AC = WC$. By Pythagorean theorem, $BC = \sqrt{144-x^2}$. Now we have $WB = WC + BC = x + \sqrt{144-x^2}$, $BX = 12$, $YA = 12$, and $AZ = x$. We can set up an equation:

$$\frac{YA}{AZ} = \frac{WB}{BX}$$

$$\frac{12}{x} = \frac{x+\sqrt{144-x^2}}{12}$$

$$144 = x^2 + x\sqrt{144-x^2}$$

$$12^2 - x^2 = x\sqrt{144-x^2}$$

$$12^4 - 2*12^2*x^2 + x^4 = 144x^2 - x^4$$

$$2x^4 - 3(12^2)x^2 + 12^4 = 0$$

$$(2x^2-144)(x^2-144) = 0$$

Solving for x, we find that $x = 6\sqrt{2}$ or $x = 12$, which we omit. The perimeter of the triangle is $12 + x + \sqrt{144-x^2}$. Plugging in $x = 6\sqrt{2}$, we get $\boxed{\textbf{(C) } 12+12\sqrt{2}}$.

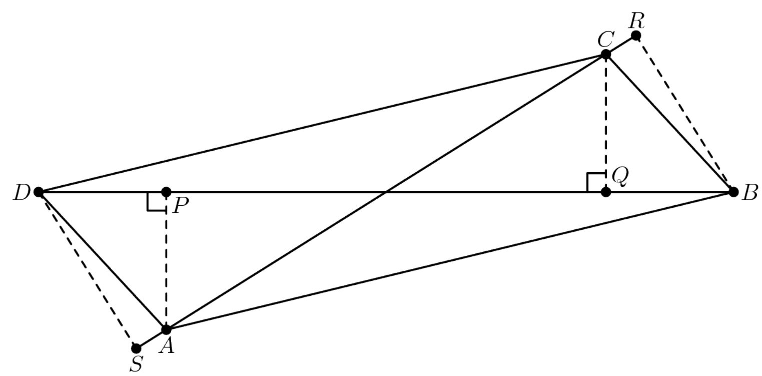
Alternatively, let $BC = S_1, AB = S_2$ and $AC = x$. Because $\frac{WB}{BX} = \frac{AY}{AZ}$, we get that $S_2^2 = S_1^2 + S_1x$. $S_1 = x$ satisfies the equation because of Pythagorean theorem, so $\triangle ABC$ is right isosceles.

3.2 Exercises

1. **(AHSME 1972)** Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. What is the diameter of this circle?

   Note that both $(39, 52)$ and $(25, 60)$ are the legs of right triangles with hypotenuse 65 ($3, 4, 5$ and $5, 12, 13$ triangles, respectively). Since the angles between these pairs of sides must be supplementary, and they must both be acute, right, or obtuse; they must both be right angles. Thus, the diameter of the circle is $\boxed{65}$.

2. **(2021 AMC 12B #24)** Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.

   Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

   Since R and S are projections of B and D, we know that $\angle BRC$ and $\angle DSA$ are both right angles.

   Since we are solving for BD, let $x = BD$.

   Let the central intersection point be M. Since the diagonals of a parallelogram bisect each other, we know that $PM = 3, MS = 4$. Also note that $\triangle AMD$ has height equal to DS and base equal to AM, and that its area is $1/4$ of the area of the parallelogram. (we can see this by viewing AD as a base, and noting that the corresponding height would be $1/2$ the corresponding height of the parallelogram, which gives us $1/4$ of the area of the parallelogram.)

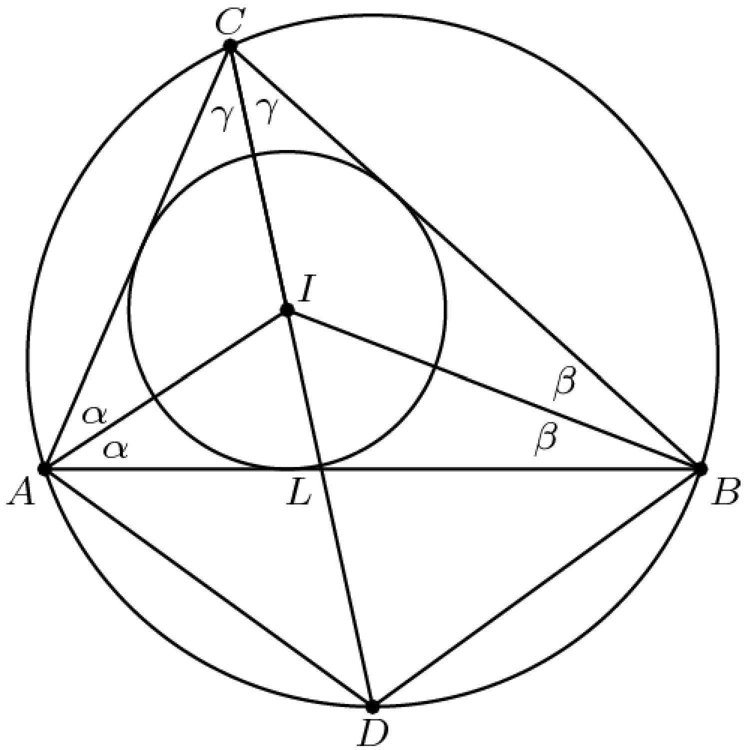
   So, the area of triangle AMD is $15/4$. Thus, $1/2 \cdot AM \cdot DS = 1/2 \cdot AM \cdot \sqrt{x^2 - 16} = 15/4$.

   Now, since DSA and APD are right angles, the quadrilateral $SAPD$ is cyclic. In particular, this allows us to use power of a point with respect to M. We have

   $$MA \cdot MS = MP \cdot MD \implies \frac{15}{2\sqrt{x^2 - 16}} \cdot 4 = 3 \cdot x \implies 100 = x^4 - 16x^2$$

   This is a quadratic equation in x^2, and by plugging in to the quadratic formula, the solutions are $8 \pm 2\sqrt{41}$. We are looking for $(2x)^2 = 4x^2 = 32 + 8\sqrt{41}$, so our final answer is $32 + 8 + 41 = \boxed{81}$.

3. **(2016 AIME I #6)** In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect $\overline{AB}$ at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



Connect D to A and D to B to form quadrilateral $ACBD$. Since quadrilateral $ACBD$ is cyclic, we can apply Ptolemy's Theorem on the quadrilateral.

Denote the length of BD and AD as z (they must be congruent, as $\angle ABD$ and $\angle DAB$ are both inscribed in arcs that have the same degree measure due to the angle bisector intersecting the circumcircle at D), and the lengths of BC, AC, AB, and CI as a, b, c, x, respectively.

After applying Ptolemy's, one will get that:

$$z(a + b) = c(x + 5)$$

Next, since $ACBD$ is cyclic, triangles ALD and CLB are similar, yielding the following equation once simplifications are made to the equation $\frac{AD}{CB} = \frac{AL}{BL}$, with the length of BL written in terms of a, b, c using the angle bisector theorem on triangle ABC:

$$zc = 3(a + b)$$

Next, drawing in the bisector of $\angle BAC$ to the incenter I, and applying the angle bisector theorem, we have that:

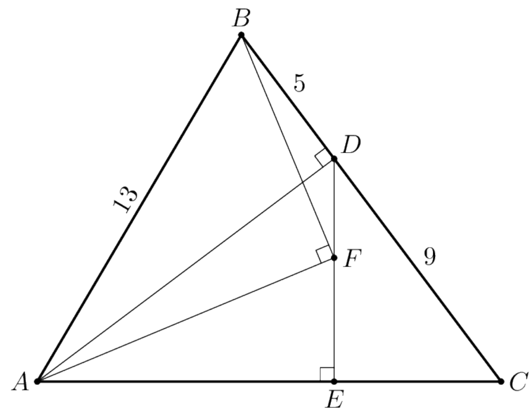
$$cx = 2(a + b)$$

Now, solving for z in the second equation, and x in the third equation and plugging them both back into the first equation, and making the substitution $w = \frac{a+b}{c}$, we get the quadratic equation:

$$3w^2 - 2w - 5 = 0$$

Solving, we get $w = 5/3$, which gives $z = 5$ and $x = 10/3$, when we rewrite the above equations in terms of w. Thus, our answer is $\boxed{013}$

4. **(2013 AMC 12B #19)** In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?



Using the similar triangles in triangle ADC gives $AE = \frac{48}{5}$ and $DE = \frac{36}{5}$. Quadrilateral $ABDF$ is cyclic, implying that $\angle B + \angle DFA = 180°$. Therefore, $\angle B = \angle EFA$, and triangles AEF and ADB are similar. Solving the resulting proportion gives $EF = 4$. Therefore, $DF = ED - EF = \frac{16}{5}$ and our answer is $\boxed{\textbf{21}}$.

4 Cyclic Quadrilateral Theorems

4.1 Examples

1. **(2022 AMC 10A #15)** Quadrilateral $ABCD$ with sides $AB = 7, BC = 24, CD = 20, DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where $a, b,$ and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

   When we look at the side lengths of the quadrilateral we see 7 and 24, which screams out 25 because of Pythagorean triplets. As a result, we can draw a line through points A and C to make a diameter of 25. See Solution 1 for a rigorous proof.

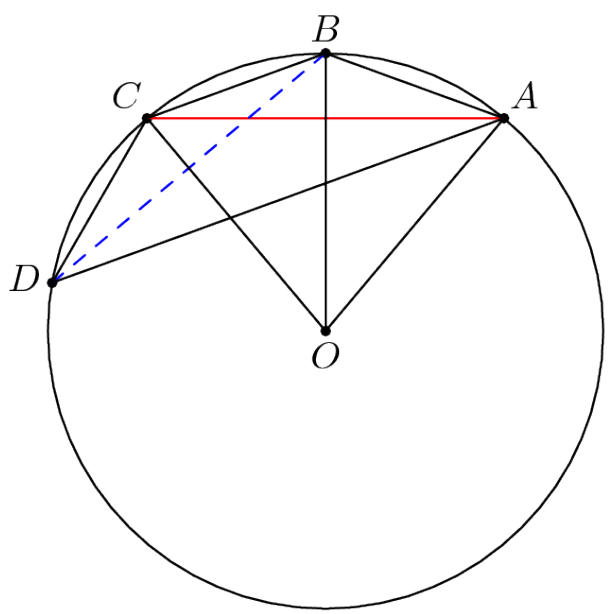
   Since the diameter is 25, we can see the area of the circle is just $\frac{625\pi}{4}$ from the formula of the area of the circle with just a diameter.

   Then we can use Brahmagupta Formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where a, b, c, d are side lengths, and s is semi-perimeter to find the area of the quadrilateral.

   If we just plug the values in, we get $\sqrt{54756} = 234$. So now the area of the region we are trying to find is $\frac{625\pi}{4} - 234 = \frac{625\pi - 936}{4}$.

   Therefore, the answer is $a + b + c = \boxed{\textbf{(D) } 1565}$.

2. **(2016 AMC 10A #24)** A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

   Let $s = 200$. Let O be the center of the circle. Then AC is twice the altitude of $\triangle OBC$ to $\overline{OB}$. Since $\triangle OBC$ is isosceles we can compute its area to be $\frac{s^2\sqrt{7}}{4}$, hence $CA = 2 \cdot \frac{2 \cdot s^2\sqrt{7}/4}{s\sqrt{2}} = s\sqrt{\frac{7}{2}}$.

   Now by Ptolemy's Theorem we have $CA^2 = s^2 + AD \cdot s \implies AD = \left(\frac{7}{2} - 1\right)s$. This gives us:

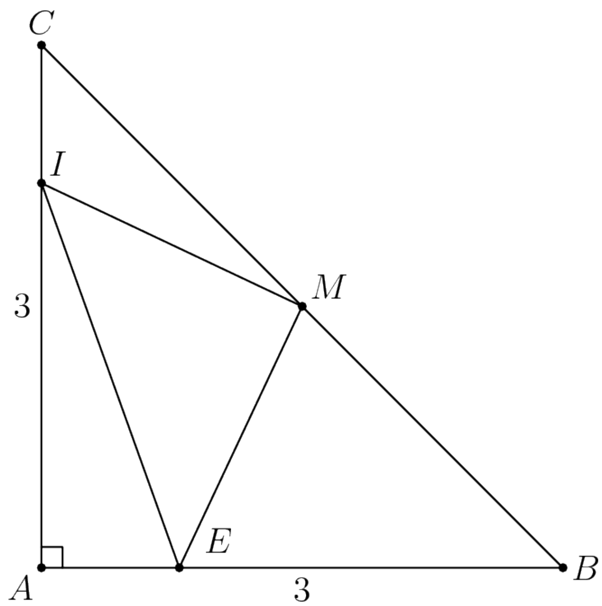
   $$\boxed{\textbf{(E) } 500.}$$

4.2 Exercises

1. **(Brahmagupta's Formula, Case 2)** Show that Brahmagupta's formula still holds when both pairs of opposite sides are parallel. (This is very simple– don't overthink it!)

   Note that the quadrilateral in this case would be a parallelogram. Opposite angles in a parallelogram are congruent, and opposite angles of a cyclic quadrilateral are supplementary. So, all the angles in the quadrilateral must be right angles, and the parallelogram is a rectangle. In particular, we have side lengths a, b, a, b around the quadrilateral, and the area is ab. Note that $a = s - b$ and $b = s - a$, so $ab = (s-a)(s-b) = \sqrt{(s-a)(s-b)(s-a)(s-b)}$, as desired.

2. **(2018 AMC 12A #20)** Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

   We first claim that $\triangle EMI$ is isosceles and right.

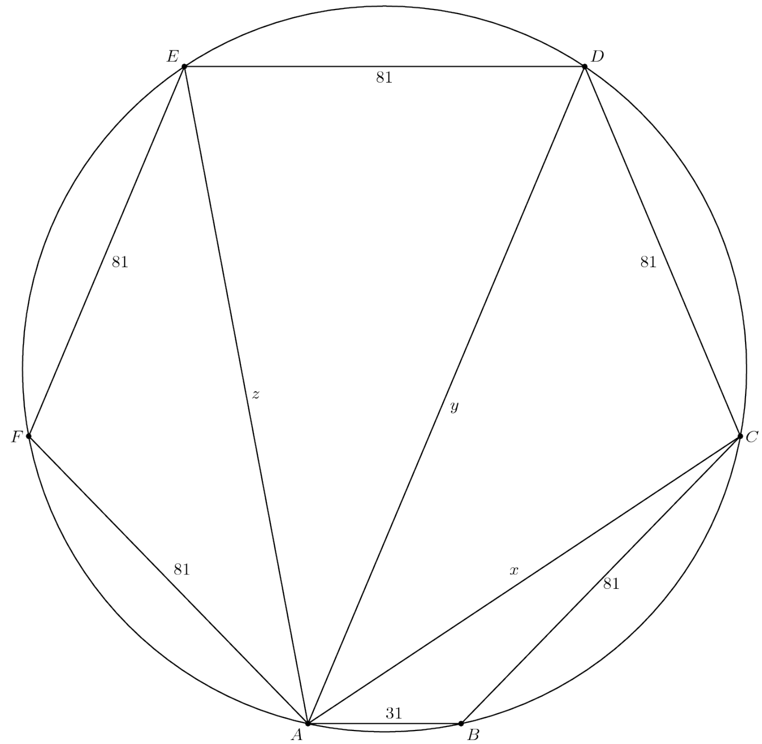
   Proof: Construct $\overline{MF} \perp \overline{AB}$ and $\overline{MG} \perp \overline{AC}$. Since $\overline{AM}$ bisects $\angle BAC$, one can deduce that $MF = MG$. Then by AAS it is clear that $MI = ME$ and therefore $\triangle EMI$ is isosceles. Since quadrilateral $AIME$ is cyclic, one can deduce that $\angle EMI = 90°$. Q.E.D.

   Since the area of $\triangle EMI$ is 2, we can find that $MI = ME = 2$, $EI = 2\sqrt{2}$

   Since M is the mid-point of $\overline{BC}$, it is clear that $AM = \frac{3\sqrt{2}}{2}$.

   Now let $AE = a$ and $AI = b$. By Ptolemy's Theorem, in cyclic quadrilateral $AIME$, we have $2a + 2b = 6$. By Pythagorean Theorem, we have $a^2 + b^2 = 8$. One can solve the simultaneous system and find $b = \frac{3+\sqrt{7}}{2}$. Then by deducting the length of $\overline{AI}$ from 3 we get $CI = \frac{3-\sqrt{7}}{2}$, giving the answer of $\boxed{\textbf{(D) } 12}$

3. **(1991 AIME #14)** A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.



Let $x = AC = BF$, $y = AD = BE$, and $z = AE = BD$.

Ptolemy's Theorem on $ABCD$ gives $81y + 31\cdot 81 = xz$, and Ptolemy on $ACDF$ gives $x\cdot z + 81^2 = y^2$. Subtracting these equations give $y^2 - 81y - 112\cdot 81 = 0$, and from this $y = 144$. Ptolemy on $ADEF$ gives $81y + 81^2 = z^2$, and from this $z = 135$. Finally, plugging back into the first equation gives $x = 105$, so $x + y + z = 105 + 144 + 135 = \boxed{384}$.

4. **(Pre-2005 Mock AIME 3 #7)** $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1, CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.

Let $BP = 3x$ and $PD = 8x$. Angle-chasing can be used to prove that $\triangle ABP \sim \triangle DCP$. Therefore $\frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{PC} = \frac{1}{4}$. This shows that $AP = 2x$ and $CP = 12x$. More angle-chasing can be used to prove that $\triangle APD \sim \triangle BPC$. This shows that $\frac{BC}{AD} = \frac{BP}{AP} = \frac{CP}{DP} = \frac{3}{2}$. It is a well-known fact that if $ABCD$ is circumscriptable around a circle then $AB + CD = AD + BC$. Therefore $BC + AD = 5$. We also know that $\frac{BC}{AD} = \frac{3}{2}$, so we can solve (algebraically or by inspection) to get that $BC = 3$ and $AD = 2$.

Brahmagupta's Formula states that the area of a cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semiperimeter and a, b, c, and d are the side lengths of the quadrilateral. Therefore the area of $ABCD$ is $\sqrt{4\cdot 3\cdot 2\cdot 1} = \sqrt{24}$. It is also a well-known fact that the area of a circumscriptable quadrilateral is sr, where r is the inradius. Therefore $5r = \sqrt{24} \Rightarrow r = \frac{\sqrt{24}}{5}$. Therefore the area of the inscribed circle is $\frac{24\pi}{25}$, and $p + q = \boxed{049}$.