ORMC AMC 10/12 Group
(More) Circles

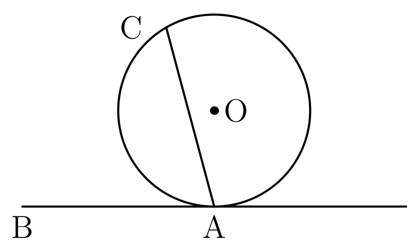
January 15, 2023

1 Warmup: Review

1. **(2020 AMC 12B # 10)** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

   Note: Last week we did this using the Inscribed Angle Theorem– this time, use Power of a Point.

2. Find the measure of $\angle BAC$ in terms of θ, where $\theta = m\overset{\frown}{AC}$:



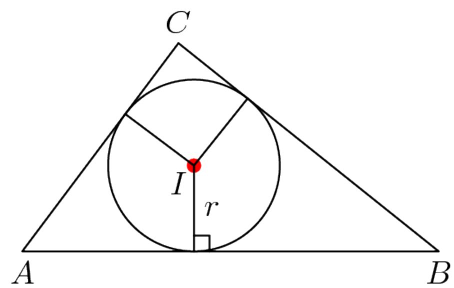
3. In unit square $ABCD$, point P is chosen at random. What is the probability that $\triangle APB$ is an obtuse triangle?

4. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.

2 Triangle Incircles and Circumcircles

The *incircle* of a triangle is the circle that sits inside the triangle and is tangent to all three sides. Its center is called the *incenter*, and is the point of intersection of the angle bisectors of the triangle. Its radius is called the *inradius* of the triangle, and is usually denoted by a lowercase r.

Consider the following figure:



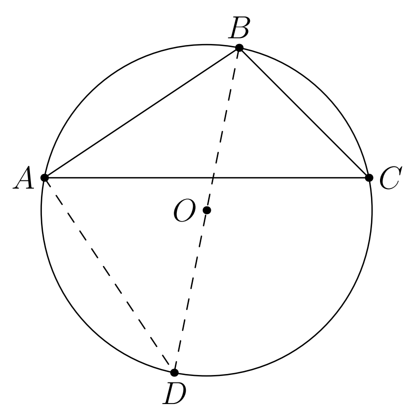
Since each side is tangent to the incircle, the inradius is perpendicular to each of the sides. So, each of the triangles $\triangle AIB, \triangle AIC, \triangle BIC$ has a base equal to one of the sides, and a height equal to the inradius. The sum of their areas is the total area, K. So, we have

$$K = \frac{1}{2}r \cdot AB + \frac{1}{2}r \cdot AC + \frac{1}{2}r \cdot BC = s \cdot r \implies \boxed{r = \frac{K}{s}}.$$

The *circumcircle* of a triangle is the circle which passes through all three vertices. Its center and radius are called the *circumcenter* and *circumradius* (R), respectively. The circumcenter is the point of intersection of the perpendicular bisectors of the triangle, and it may lie in, on, or outside the triangle

- acute triangle $\iff$ circumcenter is inside triangle
- obtuse triangle $\iff$ circumcenter is outside triangle
- right triangle $\iff$ circumcenter is the midpoint of the hypotenuse

Consider $\triangle ABC$ in the diagram below:



We draw the diameter $\overline{BD}$ through the circumcenter O, and we also draw the auxiliary line $\overline{AD}$ to create inscribed angle $\angle D$. Since $\angle C$ and $\angle D$ both intercept arc $\overset{\frown}{AB}$, we have $\angle C \cong \angle D$ by the inscribed angle theorem. Also note that $\angle DAB$ is a right angle, so $c = AB = BD\sin(D) = 2R\sin(C)$.

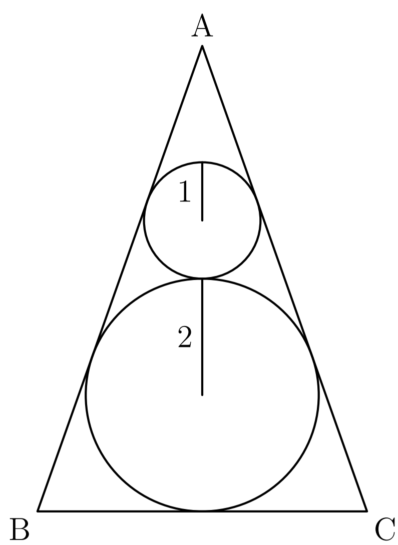
This gives us a very familiar form: $2R = \frac{c}{\sin(C)}$. If we recall the (extended) law of sines:

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)} = \frac{a}{\sin(A)} = \frac{abc}{2K} \implies \boxed{R = \frac{abc}{4K}}.$$

2.1 Examples

1. **(2004 AMC 10B #22)** A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

2. **(2006 AMC 10A #16)** A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



2.2 Exercises

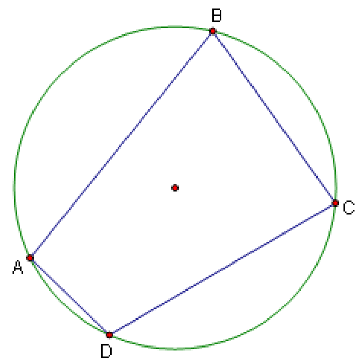
1. **(2010 AIME I #15)** In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Then $\frac{AM}{CM} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

2. **(2011 AIME II #13)** Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a+b$.

3 Cyclic Quadrilaterals

A quadrilateral that can be inscribed in a circle is called a *cyclic quadrilateral.* Note that, unlike triangles, not every quadrilateral is cyclic. This is because a circle is defined by 3 points (along its circumference), so there are relatively few places for the 4th vertex, if a quadrilateral is cyclic.

3.1 Important Properties



There are a few important properties that cyclic quadrilaterals have:

- Opposite angles are supplementary. That is, in the diagram above, $\angle A + \angle C = \angle B + \angle D = 180°$. This is because their corresponding arcs must add up to $360°$, so by the inscribed angle theorem, they are supplementary.

  The converse is true as well: if opposite angles are supplementary, then the quadrilateral is cyclic.

- The diagonals create congruent angles. Again, the key here is the inscribed angle theorem. If we include the diagonals $\overline{AC}$ and $\overline{BD}$, then the following are all true:
  - $\angle ABD = \angle ACD$
  - $\angle BCA = \angle BDA$
  - $\angle BAC = \angle BDC$
  - $\angle CAD = \angle CBD$

- The perpendicular bisectors of all the sides intersect at the circumcenter, just like in a triangle. The reason for this is the same as for triangles: every pair of vertices must be equidistant from the circumcenter, so the perpendicular bisector of any side must pass through the circumcenter.

  Note that the converse is also true: if all the perpendicular bisectors intersect in the same point, then the quadrilateral is cyclic.

3.2 Examples

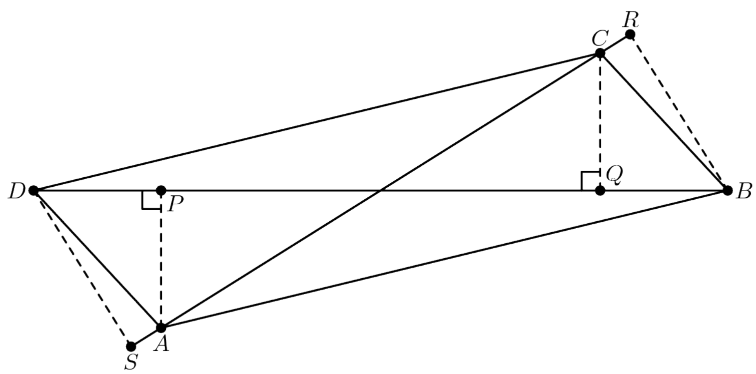
1. **(2021 Fall AMC 10A #15)** Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

2. **(2015 AMC 12B #19)** In $\triangle ABC$, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $CBWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

3.3 Exercises

1. **(AHSME 1972)** Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. What is the diameter of this circle?

2. **(2021 AMC 12B #24)** Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



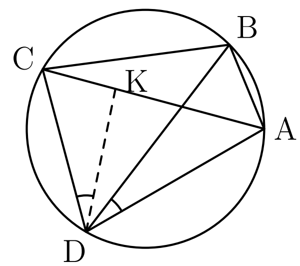
Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

3. **(2016 AIME I #6)** In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect $\overline{AB}$ at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

4. **(2013 AMC 12B #19)** In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

4 Cyclic Quadrilateral Theorems

4.1 Ptolemy's Theorem



As shown in the diagram above, begin by drawing the segment $\overline{DK}$ such that $\angle CDK \cong \angle BDA$. Note that we then also have $\angle CDB \cong \angle KDA$.

Also, by the inscribed angle theorem, we have $\angle DCK \cong \angle DBA$, and $\angle DBC \cong \angle DAK$.

So, we have 2 pairs of similar triangles: $\triangle DKC \sim \triangle DAB$ and $\triangle DBC \sim \triangle DAK$

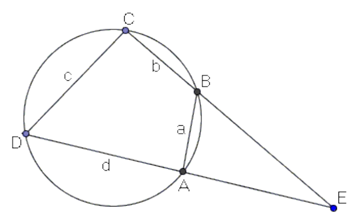
These give us the following:

$$\frac{CD}{KC} = \frac{BD}{AB} \implies AB \cdot CD = BD \cdot KC, \quad \frac{BC}{BD} = \frac{AK}{AD} \implies BC \cdot AD = BD \cdot AK$$

Adding these together gives us ptolemy's theorem:

$$\boxed{AB \cdot CD + BC \cdot AD = BD \cdot AC.}$$

4.2 Brahmagupta's Formula



We prove the case where some pair of opposite sides is not parallel. The other case is left as an exercise. Extend the non-parallel sides to meet at a point E, as shown above. Then, let $x = DE$ and $y = CE$. By Heron's Formula, $\triangle CDE$ has area $\frac{1}{4}\sqrt{(x+y+c)(x+y-c)(x-y+c)(-x+y+c)}$.

The area of triangle ABE is $\frac{a^2}{c^2}$ of this, so the cyclic quadrilateral's area is $1 - \frac{a^2}{c^2}$ of it. So, all we have to do is rewrite each of the terms in Heron's formula in terms of a, b, c, d, using the fact that $\frac{b}{x} = \frac{d}{y} = \frac{c-a}{c}$. This follows from the similarity of ABE and CDE, which we showed when proving power of a point. We get the following:

$$\begin{aligned} x+y+c &= c\frac{b+d}{c-a} + c = c\frac{b+d+c-a}{c-a} = 2c\frac{s-a}{c-a} \\ x+y-c &= c\frac{b+d}{c-a} - c = c\frac{b+d-c+a}{c-a} = 2c\frac{s-c}{c-a} \\ x-y+c &= c\frac{b-d}{c+a} + c = c\frac{b-d+c+a}{c+a} = 2c\frac{s-d}{c+a} \\ -x+y+c &= c\frac{-b+d}{c+a} + c = c\frac{-b+d+c+a}{c+a} = 2c\frac{s-b}{c+a} \end{aligned}$$

Plugging in, we find that the area of the quadrilateral is:

$$[ABCD] = \frac{c^2-a^2}{c^2}\frac{1}{4}\frac{4c^2}{c^2-a^2}\sqrt{(s-a)(s-c)(s-b)(s-d)} = \boxed{\sqrt{(s-a)(s-b)(s-c)(s-d)}.}$$

4.3 Examples

1. **(2022 AMC 10A #15)** Quadrilateral $ABCD$ with sides $AB = 7, BC = 24, CD = 20, DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

2. **(2016 AMC 10A #24)** A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

4.4 Exercises

1. **(Brahmagupta's Formula, Case 2)** Show that Brahmagupta's formula still holds when both pairs of opposite sides are parallel. (This is very simple– don't overthink it!)

2. **(2018 AMC 12A #20)** Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

3. **(1991 AIME #14)** A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

4. **(Pre-2005 Mock AIME 3 #7)** $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1, CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.