1 Warmup: Review

1. (2020 AMC 12B # 10) In unit square $ABCD$, the inscribed circle $\omega$ intersects $\overline{CD}$ at $M$, and $\overline{AM}$ intersects $\omega$ at a point $P$ different from $M$. What is $AP$?
   Note: Last week we did this using the Inscribed Angle Theorem– this time, use Power of a Point.

2. Find the measure of $\angle BAC$ in terms of $\theta$, where $\theta = m\overarc{AC}$:

   ![Diagram](image)

3. In unit square $ABCD$, point $P$ is chosen at random. What is the probability that $\triangle APB$ is an obtuse triangle?

4. Square $ABCD$ of side length 10 has a circle inscribed in it. Let $M$ be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.
2 Triangle Incircles and Circumcircles

The incircle of a triangle is the circle that sits inside the triangle and is tangent to all three sides. Its center is called the incenter, and is the point of intersection of the angle bisectors of the triangle. Its radius is called the inradius of the triangle, and is usually denoted by a lowercase $r$.

Consider the following figure:

Since each side is tangent to the incircle, the inradius is perpendicular to each of the sides. So, each of the triangles $\triangle AIB, \triangle AIC, \triangle BIC$ has a base equal to one of the sides, and a height equal to the inradius. The sum of their areas is the total area, $K$. So, we have

$$K = \frac{1}{2}r \cdot AB + \frac{1}{2}r \cdot AC + \frac{1}{2}r \cdot BC = s \cdot r \implies r = \frac{K}{s}.$$  

The circumcircle of a triangle is the circle which passes through all three vertices. Its center and radius are called the circumcenter and circumradius ($R$), respectively. The circumcenter is the point of intersection of the perpendicular bisectors of the triangle, and it may lie in, on, or outside the triangle

- acute triangle $\iff$ circumcenter is inside triangle
- obtuse triangle $\iff$ circumcenter is outside triangle
- right triangle $\iff$ circumcenter is the midpoint of the hypotenuse

Consider $\triangle ABC$ in the diagram below:

We draw the diameter $BD$ through the circumcenter $O$, and we also draw the auxiliary line $\overline{AD}$ to create inscribed angle $\angle D$. Since $\angle C$ and $\angle D$ both intercept arc $\overline{AB}$, we have $\angle C \cong \angle D$ by the inscribed angle theorem. Also note that $\angle DAB$ is a right angle, so $c = AB = BD \sin(D) = 2R \sin(C)$.

This gives us a very familiar form: $2R = \frac{c}{\sin(C)}$. If we recall the (extended) law of sines:

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)} = \frac{a}{\sin(A)} = \frac{abc}{2K} \implies R = \frac{abc}{4K}.$$  


2.1 Examples

1. (2004 AMC 10B #22) A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

2. (2006 AMC 10A #16) A circle of radius 1 is tangent to a circle of radius 2. The sides of \( \triangle ABC \) are tangent to the circles as shown, and the sides \( AB \) and \( AC \) are congruent. What is the area of \( \triangle ABC \)?

2.2 Exercises

1. (2010 AIME I #15) In \( \triangle ABC \) with \( AB = 12, BC = 13, \) and \( AC = 15 \), let \( M \) be a point on \( AC \) such that the incircles of \( \triangle ABM \) and \( \triangle BCM \) have equal radii. Then \( \frac{AM}{CM} = \frac{p}{q} \), where \( p \) and \( q \) are relatively prime positive integers. Find \( p + q \).

2. (2011 AIME II #13) Point \( P \) lies on the diagonal \( AC \) of square \( ABCD \) with \( AP > CP \). Let \( O_1 \) and \( O_2 \) be the circumcenters of triangles \( ABP \) and \( CDP \) respectively. Given that \( AB = 12 \) and \( \angle O_1PO_2 = 120^\circ \), then \( AP = \sqrt{a} + \sqrt{b} \), where \( a \) and \( b \) are positive integers. Find \( a + b \).
3 Cyclic Quadrilaterals

A quadrilateral that can be inscribed in a circle is called a cyclic quadrilateral. Note that, unlike triangles, not every quadrilateral is cyclic. This is because a circle is defined by 3 points (along its circumference), so there are relatively few places for the 4th vertex, if a quadrilateral is cyclic.

3.1 Important Properties

There are a few important properties that cyclic quadrilaterals have:

- Opposite angles are supplementary. That is, in the diagram above, $\angle A + \angle C = \angle B + \angle D = 180^\circ$. This is because their corresponding arcs must add up to 360°, so by the inscribed angle theorem, they are supplementary.

The converse is true as well: if opposite angles are supplementary, then the quadrilateral is cyclic.

- The diagonals create congruent angles. Again, the key here is the inscribed angle theorem. If we include the diagonals $\overline{AC}$ and $\overline{BD}$, then the following are all true:
  - $\angle ABD = \angle ACD$
  - $\angle BCA = \angle BDA$
  - $\angle BAC = \angle BDC$
  - $\angle CAD = \angle CBD$

- The perpendicular bisectors of all the sides intersect at the circumcenter, just like in a triangle. The reason for this is the same as for triangles: every pair of vertices must be equidistant from the circumcenter, so the perpendicular bisector of any side must pass through the circumcenter.

Note that the converse is also true: if all the perpendicular bisectors intersect in the same point, then the quadrilateral is cyclic.

3.2 Examples

1. (2021 Fall AMC 10A #15) Isosceles triangle $ABC$ has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line $AB$ at $B$ and to line $AC$ at $C$. What is the area of the circle that passes through vertices $A$, $B$, and $C$?

2. (2015 AMC 12B #19) In $\triangle ABC$, $\angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $CBWZ$ are constructed outside of the triangle. The points $X$, $Y$, $Z$, and $W$ lie on a circle. What is the perimeter of the triangle?
3.3 Exercises

1. **(AHSME 1972)** Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. What is the diameter of this circle?

2. **(2021 AMC 12B #24)** Let $ABCD$ be a parallelogram with area 15. Points $P$ and $Q$ are the projections of $A$ and $C$, respectively, onto the line $BD$; and points $R$ and $S$ are the projections of $B$ and $D$, respectively, onto the line $AC$. See the figure, which also shows the relative locations of these points.

![Diagram of parallelogram and projections](image)

Suppose $PQ = 6$ and $RS = 8$, and let $d$ denote the length of $BD$, the longer diagonal of $ABCD$. Then $d^2$ can be written in the form $m + n\sqrt{p}$, where $m$, $n$, and $p$ are positive integers and $p$ is not divisible by the square of any prime. What is $m + n + p$?

3. **(2016 AIME I #6)** In $\triangle ABC$ let $I$ be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect $AB$ at $L$. The line through $C$ and $L$ intersects the circumscribed circle of $\triangle ABC$ at the two points $C$ and $D$. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

4. **(2013 AMC 12B #19)** In triangle $ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points $D$, $E$, and $F$ lie on segments $BC$, $CA$, and $DE$, respectively, such that $AD \perp BC$, $DE \perp AC$, and $AF \perp BF$. The length of segment $DF$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?
4 Cyclic Quadrilateral Theorems

4.1 Ptolemy’s Theorem

As shown in the diagram above, begin by drawing the segment \(DK\) such that \(\angle CDK \cong \angle BDA\). Note that we then also have \(\angle CDB \cong \angle KDA\). Also, by the inscribed angle theorem, we have \(\angle DCK \cong \angle DBA\), and \(\angle DBC \cong \angle DAK\). So, we have 2 pairs of similar triangles: \(\triangle DKC \sim \triangle DAB\) and \(\triangle DBC \sim \triangle DAK\). These give us the following:

\[
\frac{CD}{KC} = \frac{BD}{AB} \implies AB \cdot CD = BD \cdot KC, \quad \frac{BC}{BD} = \frac{AK}{AD} \implies BC \cdot AD = BD \cdot AK
\]

Adding these together gives us Ptolemy’s theorem:

\[AB \cdot CD + BC \cdot AD = BD \cdot AC.\]

4.2 Brahmagupta’s Formula

We prove the case where some pair of opposite sides is not parallel. The other case is left as an exercise. Extend the non-parallel sides to meet at a point \(E\), as shown above. Then, let \(x = DE\) and \(y = CE\). By Heron’s Formula, \(\triangle CDE\) has area \(\frac{1}{4} \sqrt{(x + y + c)(x + y - c)(x - y + c)(-x + y + c)}\).

The area of triangle \(ABE\) is \(\frac{a^2}{c^2}\) of this, so the cyclic quadrilateral’s area is \(1 - \frac{a^2}{c^2}\) of it. So, all we have to do is rewrite each of the terms in Heron’s formula in terms of \(a, b, c, d\), using the fact that \(\frac{b}{c} = \frac{d}{c} = \frac{a}{c}\). This follows from the similarity of \(ABE\) and \(CDE\), which we showed when proving power of a point. We get the following:

\[
\begin{align*}
x + y + c &= \frac{b + d}{c-a} + c = \frac{b + d + c - a}{c-a} = 2c \frac{s - a}{c-a} \\
x + y - c &= \frac{b + d}{c-a} - c = \frac{b + d - c + a}{c-a} = 2c \frac{s - c}{c-a} \\
x - y + c &= \frac{b - d}{c+a} + c = \frac{b - d + c + a}{c+a} = 2c \frac{s - d}{c+a} \\
x + y - c &= \frac{-b + d}{c+a} + c = \frac{-b + d + c + a}{c+a} = 2c \frac{s - b}{c+a}
\end{align*}
\]

Plugging in, we find that the area of the quadrilateral is:

\[
[ABCD] = \frac{c^2 - a^2}{c^2} \frac{4c^2}{4c^2 - a^2} \sqrt{(s-a)(s-c)(s-b)(s-d)} = \frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{c^2}.
\]
4.3 Examples

1. (2022 AMC 10A #15) Quadrilateral $ABCD$ with sides $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where $a$, $b$, and $c$ are positive integers such that $a$ and $c$ have no common prime factor. What is $a + b + c$?

2. (2016 AMC 10A #24) A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

4.4 Exercises

1. (Brahmagupta’s Formula, Case 2) Show that Brahmagupta’s formula still holds when both pairs of opposite sides are parallel. (This is very simple—don’t overthink it!)

2. (2018 AMC 12A #20) Triangle $ABC$ is an isosceles right triangle with $AB = AC = 3$. Let $M$ be the midpoint of hypotenuse $BC$. Points $I$ and $E$ lie on sides $AC$ and $AB$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle $EMI$ has area 2, the length $CI$ can be written as $a - \sqrt{b}$, where $a$, $b$, and $c$ are positive integers and $b$ is not divisible by the square of any prime. What is the value of $a + b + c$?

3. (1991 AIME #14) A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $AB$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from $A$.

4. (Pre-2005 Mock AIME 3 #7) $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at $P$. If $AB = 1$, $CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where $p$ and $q$ are relatively prime positive integers. Determine $p + q$. 