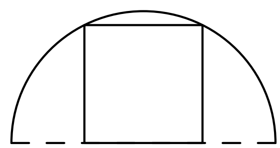
ORMC AMC 10/12 Group
Circles

January 8, 2023

1 Warmup: Circle Basics

1.1 Exercises

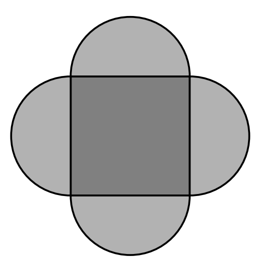
1. **(2006 AMC 10B #8)** A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



Since the area of the square is 40, the length of a side is $\sqrt{40} = 2\sqrt{10}$. The distance between the center of the semicircle and one of the bottom vertices of the square is half the length of the side, which is $\sqrt{10}$.

Using the Pythagorean Theorem to find the radius r of the semicircle, $r^2 = (2\sqrt{10})^2 + (\sqrt{10})^2 = 50$. So, the area of the semicircle is $\frac{1}{2} \cdot \pi \cdot 50 = \boxed{25\pi}$

2. **(2006 AMC 10B #6)** A region is bounded by semicircular arcs constructed on the side of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region?

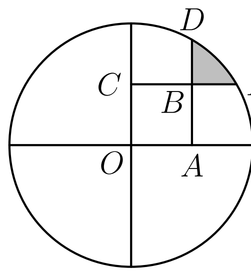


Since the side of the square is the diameter of the semicircle, the radius of the semicircle is $\frac{1}{2} \cdot \frac{2}{\pi} = \frac{1}{\pi}$.

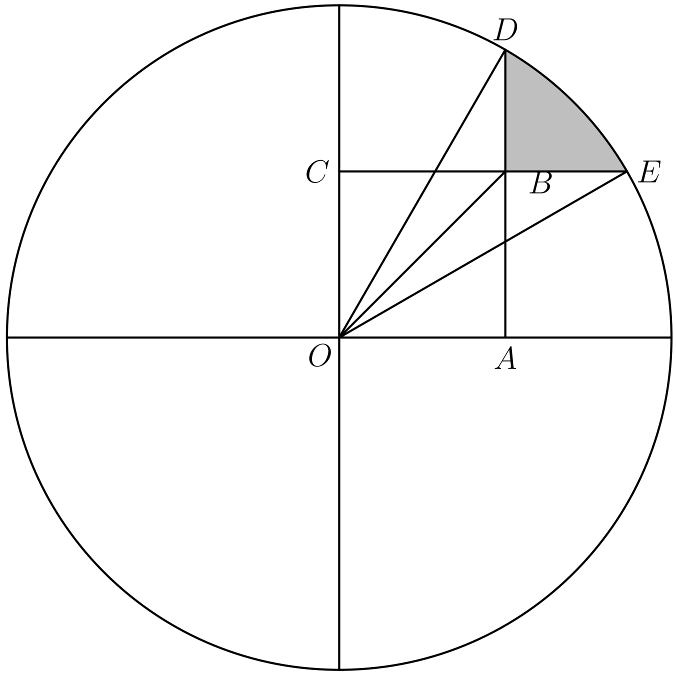
Since the length of one of the semicircular arcs is half the circumference of the corresponding circle, the length of one arc is $\frac{1}{2} \cdot 2 \cdot \pi \cdot \frac{1}{\pi} = 1$.

Since the desired perimeter is made up of four of these arcs, the perimeter is $4 \cdot 1 = \boxed{4}$.

3. **(2006 AMC 10B #19)** A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?

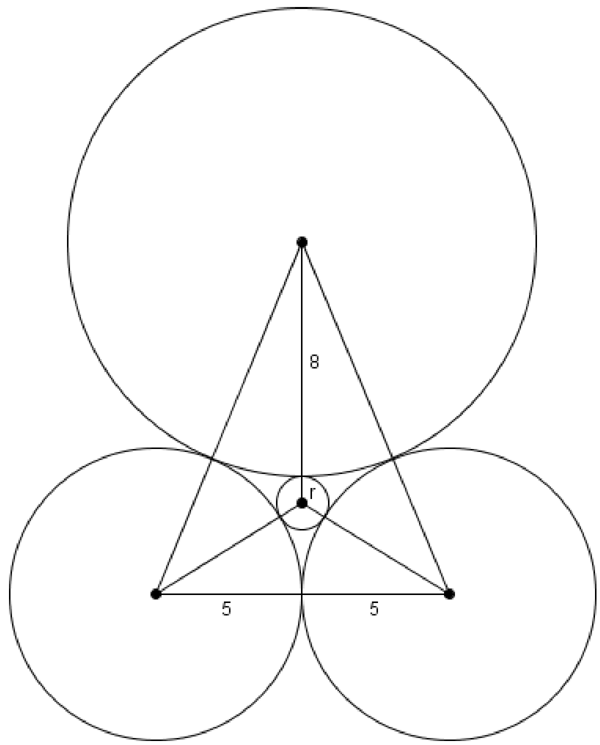


From the Pythagorean Theorem, we can see that DA is $\sqrt{3}$. Then, $DB = DA - BA = \sqrt{3} - 1$. The area of the shaded element is the area of sector DOE minus the areas of triangle DBO and triangle EBO combined. Below is an image to help.



Using the Base Altitude formula, where DB and BE are the bases and OA and CO are the altitudes, respectively, $[DBO] = [EBO] = \frac{\sqrt{3}-1}{2}$. The area of sector DOE is $\frac{1}{12}$ of circle O. The area of circle O is 4π, and therefore we have the area of sector DBE to be $\boxed{\frac{\pi}{3} + 1 - \sqrt{3}}$.

4. **(1997 AIME #4)** Circles of radii $5, 5, 8$, and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.



If (in the diagram above) we draw the line going through the centers of the circles with radii 8 and $\frac{m}{n} = r$, that line is the perpendicular bisector of the segment connecting the centers of the two circles with radii 5. Then we form two right triangles, of lengths $5, x, 5+r$ and $5, 8+r+x, 13$, wher x is the distance between the center of the circle in question and the segment connecting the centers of the two circles of radii 5. By the Pythagorean Theorem, we now have two equations with two unknowns:

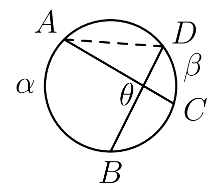
$$
\begin{aligned}
5^2 + x^2 &= (5+r)^2 \\
x &= \sqrt{10r + r^2} \\
\\
(8 + r + \sqrt{10r + r^2})^2 + 5^2 &= 13^2 \\
8 + r + \sqrt{10r + r^2} &= 12 \\
\sqrt{10r + r^2} &= 4 - r \\
10r + r^2 &= 16 - 8r + r^2 \\
r &= \frac{8}{9}
\end{aligned}
$$

So $m + n = \boxed{17}$.

2 Inscribed Angle Theorem

2.1 Examples

1. **(Intersecting Chords Theorem: Case 1)** Find the angle θ in terms of α and β.



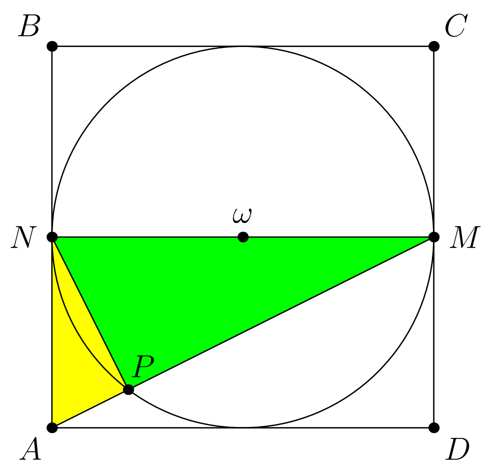
We draw the auxiliary line $\overline{AD}$ to create inscribed angles $\angle A$ and $\angle D$. By the inscribed angle theorem, the measures of these angles are $\beta/2$ and $\alpha/2$, respectively. And, by the exterior angle theorem for triangles,

$$\theta = \boxed{\frac{\alpha+\beta}{2}}.$$

2. **(2020 AMC 12B # 10)** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

Let N be the midpoint of $\overline{AB}$, from which $\angle ANM = 90°$. Note that $\angle NPM = 90°$ by the Inscribed Angle Theorem.

We have the following diagram:



Since $AN = \frac{1}{2}$ and $NM = 1$, we get $AM = \frac{\sqrt{5}}{2}$ by the Pythagorean Theorem.

Let $AP = x$. It follows that $PM = \frac{\sqrt{5}}{2} - x$. Applying the Pythagorean Theorem to right $\triangle ANP$ gives $NP^2 = \left(\frac{1}{2}\right)^2 - x^2$, and applying the Pythagorean Theorem to right $\triangle MNP$ gives $NP^2 = 1^2 - \left(\frac{\sqrt{5}}{2} - x\right)^2$. Equating the expressions for NP^2 produces

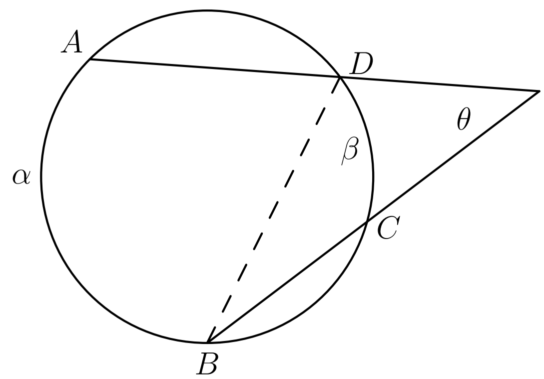
$$\left(\frac{1}{2}\right)^2 - x^2 = 1^2 - \left(\frac{\sqrt{5}}{2} - x\right)^2 \implies \frac{1}{4} - x^2 = 1 - \frac{5}{4} + \sqrt{5}x - x^2 \implies \frac{1}{2} = \sqrt{5}x.$$

Finally, dividing both sides by $\sqrt{5}$ and then rationalizing the denominator, we obtain

$$x = \frac{1}{2\sqrt{5}} = \boxed{\textbf{(B) } \frac{\sqrt{5}}{10}}.$$

2.2 Exercises

1. **(Intersecting Chords Theorem: Case 2)** Find the angle θ in terms of α and β.



   We start by drawing the auxiliary line $\overline{BD}$. This creates inscribed angles $\angle B$ and $\angle ADB$. Again by the triangle exterior angle theorem, we have

$$\theta + \angle B = \angle ADB \implies \boxed{\theta = \frac{\alpha - \beta}{2}}.$$

2. **(2014 AMC 12A #12)** Two circles intersect at points A and B. The minor arcs AB measure $30°$ on one circle and $60°$ on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

   Let the radius of the smaller and larger circles be r and R, respectively. Also, let their centers be O_1 and O_2, respectively. Now draw two congruent chords from points A and B to the end of the smaller circle, creating an isosceles triangle. Label that point X. Recalling the Inscribed Angle Theorem, we then see that $m\angle AXB = \frac{m\angle AO_1B}{2} = 30° = m\angle AO_2B$. Based on this information, we can conclude that triangles AXB and AO_2B are congruent via ASA Congruence.

   Next draw the height of AXB from X to AB. Note we've just created a right triangle with hypotenuse R, base $\frac{r}{2}$, and height $\frac{r\sqrt{3}}{2} + r$ Thus using the Pythagorean Theorem we can express R^2 in terms of r

$$R^2 = (\frac{r}{2})^2 + (\frac{r\sqrt{3}}{2} + r)^2 = r^2 + \frac{r^2}{4} + \frac{3r^2}{4} + (2)(\frac{r\sqrt{3}}{2})(r) = 2r^2 + r^2\sqrt{3} = r^2(2 + \sqrt{3})$$

   We can now determine the ratio between the larger and smaller circles:

$$\frac{Area[O_2]}{Area[O_1]} = \frac{\pi R^2}{\pi r^2} = \frac{\pi r^2(2 + \sqrt{3})}{\pi r^2} = \boxed{2 + \sqrt{3}}$$

3. **(2011 AMC 10B #17)** In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?

Note that $\overset{\frown}{BE}$ intercepts $\angle BAE$. Since, $\overset{\frown}{BE} = 180$, thus $\angle BAE = 90$ (courtesy of the Inscribed Angles Theorem).

Since we know that $\angle BAE = 90$, then $\angle AEB + \angle ABE = 90$, (courtesy of the Triangle Sum Theorem) and also $5\angle AEB = 4\angle ABE$. By solving this variation, $\angle AEB = 40$ and $\angle ABE = 50$. After that, due to the Alternate Interior Angles Theorem, $\angle ABE \cong \angle BED$, which means $\angle BED = 50$.

After doing some angle chasing, then these following facts should be true, $\overset{\frown}{AB} = 80$ $\overset{\frown}{BD} = 100$ $\overset{\frown}{AE} = 100$.

Note that the arcs have to equal 360, so, $360 = \overset{\frown}{AB} + \overset{\frown}{BD} + \overset{\frown}{DE} + \overset{\frown}{AE}$

$360 = 80 + 100 + 100 + \overset{\frown}{DE}$

$\overset{\frown}{DE} = 80$

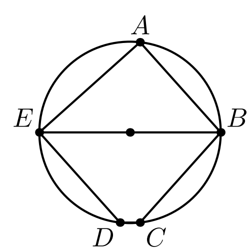
Notice how $\overset{\frown}{DB}$ intercepts $\angle BCD$ and that $\overset{\frown}{DB} = \overset{\frown}{DE} + \overset{\frown}{AE} + \overset{\frown}{AB}$.

$\overset{\frown}{DB} = 80 + 100 + 80$

$\overset{\frown}{DB} = 260$

According to the Inscribed Angles Theorem, $2\angle BCD = \overset{\frown}{DB}$, therefore the answer is

$$\frac{260}{2} = \boxed{130}$$



3 Power of a Point

3.1 Examples

1. **(ARML)** In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

   Let $AR = x, BR = 4x, CR = 4y, DR = 9y$. Then, we are looking for x/y. By power of a point, we have $4x^2 = 36y^2 \implies (x/y)^2 = 36/4 = 9 \implies x/y = \boxed{3}$.

2. Two tangents from an external point P are drawn to a circle and intersect it at A and B. A third tangent meets the circle at T, and the tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at points Q and R, respectively (this means that T is on the minor arc AB). If $AP = 20$, find the perimeter of $\triangle PQR$.

   Let $a = AQ, b = BR$. It follows from power of a point that any two tangents have the same length. So, $AP = BP = 20$ and $PQ = 20 - a, PR = 20 - b$. And, $QT = a$, $RT = b$. So, the perimeter of the triangle is $\boxed{40}$.

3.2 Exercises

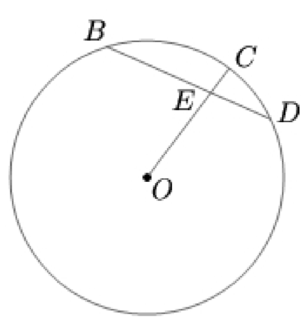
1. **(ARML)** Chords AB and CD of a given circle are perpendicular to each other and intersect at a right angle at point E. Given that $BE = 16$, $DE = 4$, and $AD = 5$, find CE.

   Note that AD is the hypotenuse of right triangle $\triangle AED$, so we have $AE = 3$. Then, by power of a point, $4 \cdot CE = 16 \cdot 3 \implies \boxed{CE = 12}$.

2. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.

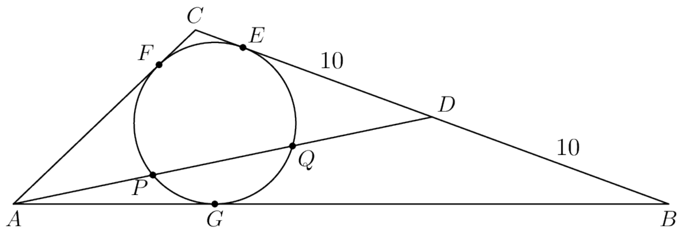
   By the pythagorean theorem, $XM = 5\sqrt{5}$. Then, label point P as the intersection between CM and the circle. Let $x = PC$. By power of a point, we have $5\sqrt{5} \cdot x = 5^2$ (since the tangent from point C to the circle has length 5. This means that $\boxed{x = \sqrt{5}}$.

3. **(1971 CMO #1)** DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given $EC = 1$, find the radius of the circle.



   Extend the radius to a diameter. Then, by power of a point, we have $5 \cdot 3 = 1 \cdot (d-1) = d - 1 \implies d = 16 \implies \boxed{r = 8}$.

4. **(2005 AIME I #15)** Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.



Let E, F and G be the points of tangency of the incircle with BC, AC and AB, respectively. Without loss of generality, let $AC < AB$, so that E is between D and C. Let the length of the median be $3m$. Then by two applications of the Power of a Point Theorem, $DE^2 = 2m \cdot m = AF^2$, so $DE = AF$. Now, CE and CF are two tangents to a circle from the same point, so by the Two Tangent Theorem $CE = CF = c$ and thus $AC = AF + CF = DE + CE = CD = 10$. Then $DE = AF = AG = 10 - c$ so $BG = BE = BD + DE = 20 - c$ and thus $AB = AG + BG = 30 - 2c$.

Now, by Stewart's Theorem in triangle $\triangle ABC$ with cevian $\overline{AD}$, we have

$$(3m)^2 \cdot 20 + 20 \cdot 10 \cdot 10 = 10^2 \cdot 10 + (30 - 2c)^2 \cdot 10.$$

Our earlier result from Power of a Point was that $2m^2 = (10 - c)^2$, so we combine these two results to solve for c and we get

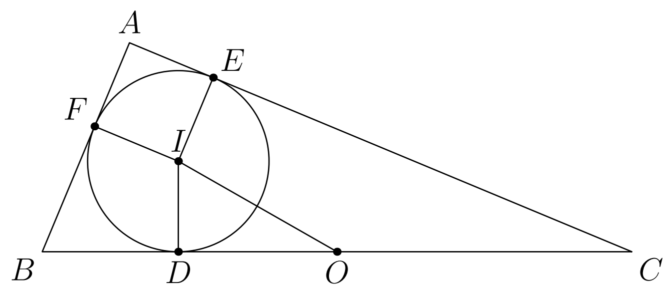
$$9(10 - c)^2 + 200 = 100 + (30 - 2c)^2 \quad \Longrightarrow \quad c^2 - 12c + 20 = 0.$$

Thus $c = 2$ or $= 10$. We discard the value $c = 10$ as extraneous (it gives us a line) and are left with $c = 2$, so our triangle has area $\sqrt{28 \cdot 18 \cdot 8 \cdot 2} = 24\sqrt{14}$ and so the answer is $24 + 14 = \boxed{038}$.

4 Triangle Incircles and Circumcircles

4.1 Examples

1. **(2004 AMC 10B #22)** A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?



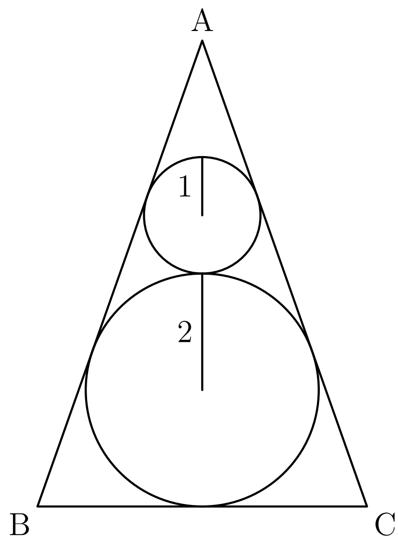
This is a right triangle. Pick a coordinate system so that the right angle is at $(0,0)$ and the other two vertices are at $(12,0)$ and $(0,5)$.

As this is a right triangle, the center of the circumcircle is in the middle of the hypotenuse, at $(6,2.5)$.

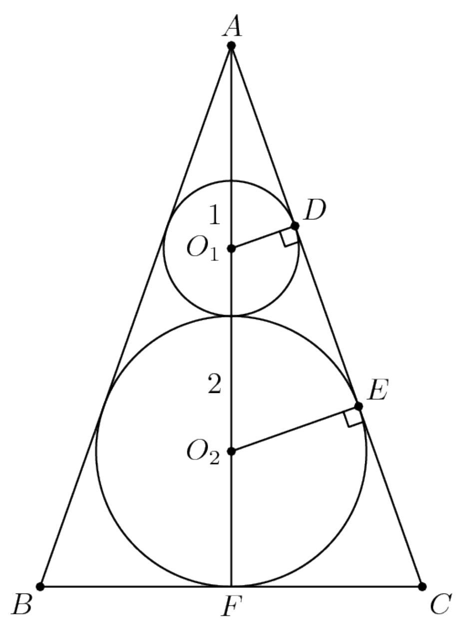
The radius r of the inscribed circle can be computed using the well-known identity $\frac{rP}{2}=S$, where S is the area of the triangle and P its perimeter. In our case, $S=\frac{5\cdot 12}{2}=30$ and $P=5+12+13=30$. Thus, $r=2$. As the inscribed circle touches both legs, its center must be at $(r,r)=(2,2)$.

The distance of these two points is then $\sqrt{(6-2)^2+(2.5-2)^2}=\sqrt{16.25}=\sqrt{\frac{65}{4}}=\boxed{\frac{\sqrt{65}}{2}}$.

2. **(2006 AMC 10A #16)** A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



Let the centers of the smaller and larger circles be O_1 and O_2 , respectively. Let their tangent points to $\triangle ABC$ be D and E, respectively. We can then draw the following diagram:



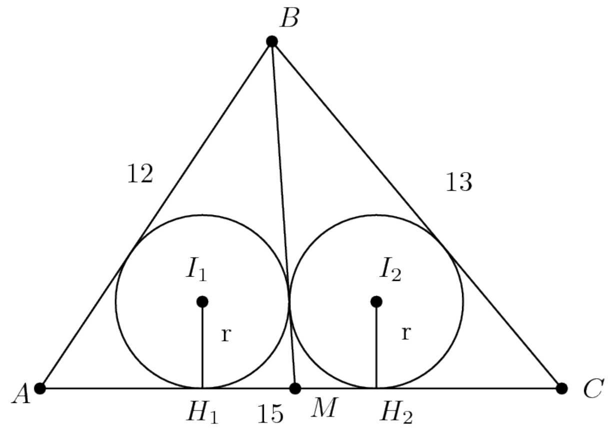
We see that $\triangle ADO_1 \sim \triangle AEO_2 \sim \triangle AFC$. Using the first pair of similar triangles, we write the proportion:
$\frac{AO_1}{AO_2} = \frac{DO_1}{EO_2} \implies \frac{AO_1}{AO_1+3} = \frac{1}{2} \implies AO_1 = 3$ By the Pythagorean Theorem, we have $AD = \sqrt{3^2 - 1^2} = \sqrt{8}$.

Now using $\triangle ADO_1 \sim \triangle AFC$,
$\frac{AD}{AF} = \frac{DO_1}{FC} \implies \frac{2\sqrt{2}}{8} = \frac{1}{FC} \implies FC = 2\sqrt{2}$ Hence, the area of the triangle is

$$\frac{1}{2} \cdot AF \cdot BC = \frac{1}{2} \cdot AF \cdot (2 \cdot CF) = AF \cdot CF = 8\left(2\sqrt{2}\right) = \boxed{\textbf{(D) } 16\sqrt{2}}$$

4.2 Exercises

1. **(2010 AIME I #15)** In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Then $\frac{AM}{CM} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



Let $AM = x$, then $CM = 15 - x$. Also let $BM = d$ Clearly, $\frac{[ABM]}{[CBM]} = \frac{x}{15-x}$. We can also express each area by the rs formula. Then $\frac{[ABM]}{[CBM]} = \frac{p(ABM)}{p(CBM)} = \frac{12+d+x}{28+d-x}$. Equating and cross-multiplying yields $25x + 2dx = 15d + 180$ or $d = \frac{25x-180}{15-2x}$. Note that for d to be positive, we must have $7.2 < x < 7.5$.

By Stewart's Theorem, we have $12^2(15-x)+13^2x = d^2 15+15x(15-x)$ or $432 = 3d^2+40x-3x^2$. Brute forcing by plugging in our previous result for d, we have $432 = \frac{3(25x-180)^2}{(15-2x)^2} + 40x - 3x^2$. Clearing the fraction and gathering like terms, we get $0 = 12x^4 - 340x^3 + 2928x^2 - 7920x$.

Aside: Since x must be rational in order for our answer to be in the desired form, we can use the Rational Root Theorem to reveal that $12x$ is an integer. The only such x in the above-stated range is $\frac{22}{3}$.

Legitimately solving that quartic, note that $x = 0$ and $x = 15$ should clearly be solutions, corresponding to the sides of the triangle and thus degenerate cevians. Factoring those out, we get $0 = 4x(x-15)(3x^2-40x+132) = x(x-15)(x-6)(3x-22)$. The only solution in the desired range is thus $\frac{22}{3}$. Then $CM = \frac{23}{3}$, and our desired ratio $\frac{AM}{CM} = \frac{22}{23}$, giving us an answer of $\boxed{045}$.

2. **(2011 AIME II #13)** Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Denote the midpoint of $\overline{DC}$ be E and the midpoint of $\overline{AB}$ be F. Because they are the circumcenters, both Os lie on the perpendicular bisectors of AB and CD and these bisectors go through E and F.

It is given that $\angle O_1PO_2 = 120°$. Because O_1P and O_1B are radii of the same circle, the have the same length. This is also true of O_2P and O_2D. Because $m\angle CAB = m\angle ACD = 45°$, $m\overset{\frown}{PD} = m\overset{\frown}{PB} = 2(45°) = 90°$. Thus, O_1PB and O_2PD are isosceles right triangles. Using the given information above and symmetry, $m\angle DPB = 120°$. Because ABP and ADP share one side, have one side with the same length, and one equal angle, they are congruent by SAS. This is also true for triangle CPB and CPD. Because angles APB and APD are equal and they sum to 120 degrees, they are each 60 degrees. Likewise, both angles CPB and CPD have measures of 120 degrees.

Because the interior angles of a triangle add to 180 degrees, angle ABP has measure 75 degrees and angle PDC has measure 15 degrees. Subtracting, it is found that both angles O_1BF and O_2DE have measures of 30 degrees. Thus, both triangles O_1BF and O_2DE are 30-60-90 right triangles. Because F and E are the midpoints of AB and CD respectively, both FB and DE have lengths of 6. Thus, $DO_2 = BO_1 = 4\sqrt{3}$. Because of 45-45-90 right triangles, $PB = PD = 4\sqrt{6}$.

Now, letting $x = AP$ and using Law of Cosines on $\triangle ABP$, we have

$$96 = 144 + x^2 - 24x\frac{\sqrt{2}}{2}$$
$$0 = x^2 - 12x\sqrt{2} + 48$$

Using the quadratic formula, we arrive at

$$x = \sqrt{72} \pm \sqrt{24}$$

Taking the positive root, $AP = \sqrt{72} + \sqrt{24}$ and the answer is thus $\boxed{096.}$