ORMC Olympiad Group
Winter: Week 1
Analysis: Functions

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Problems

1. Determine if the following functions are 1-1, onto, or bijective?
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
   (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$
   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$
   (d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$
   (e) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$
   (f) $f : \mathbb{Z} \to \mathbb{Z}^+$, $f(x) = |x| + 1$
   (g) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 5x$
2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + y) = f(x) + f(y)$.
3. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) = f(x) + f(y)$.
4. **TJNMO-FR 2017** Let f be a real valued function which satisfies the following equality

$$f(xy - x - y + 1) = f(xy) + f(x) - f(y) + 6x - 3$$

for any real values x, y. If $f(\frac{1}{3}) = 1$, find $f(-33)$.

5. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$.
6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x + y) = f(x) + f(y)$.
7. **(PSS Functions P4)** Prove that function f is periodic if for some $a > 0$ and all $x \in \mathbb{R}$
$$f(x + a) = \frac{1 + f(x)}{1 - f(x)}$$
8. **(IMOmath P3)** Function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x + f(x)$ for all $x \in \mathbb{R}$. Find all solutions of the equation $f(f(x)) = 0$
9. **(IMOmath P1)** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x + y) + 1$.
10. **(Turkish JNMO 2000 Second Round P3)** $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation
$$f(x)f(y) - af(xy) = x + y$$
for every real numbers x, y. Find all possible real values of a.
11. **(Turkish NMO 2004 Second Round P4)** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the condition $f(n) - f(n + f(m)) = m$ for all $m, n \in \mathbb{Z}$
12. **(Turkish NMO 2008 Second Round P4)** $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ satisfy the given conditions

    $a)$ $f(0, 0) = 1$, $f(0, 1) = 1$,

    $b)$ $\forall k \notin \{0, 1\}$ $f(0, k) = 0$ and

    $c)$ $\forall n \geq 1$ and k , $f(n, k) = f(n - 1, k) + f(n - 1, k - 2n)$

    find the sum $\displaystyle\sum_{k=0}^{\binom{2009}{2}} f(2008, k)$