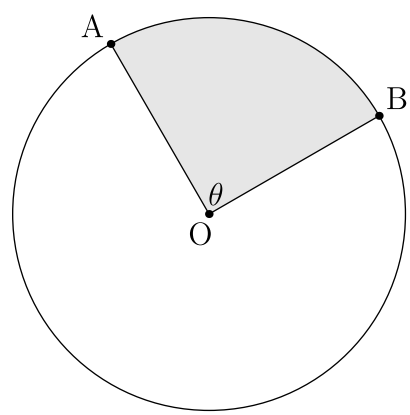
ORMC AMC 10/12 Group
Circles

January 8, 2023

1 Warmup: Circle Basics

As usual, we will denote the radius of a circle by r, diameter by d, circumference by C, and area by A. Recall that: $d = 2r,\ C = 2\pi r = \pi d,\ A = \pi r^2$.

A *central angle* θ of a circle is the angle created by two radii. The portion of the circumference created by a central angle is called an *arc*. The area created by a central angle is called a *sector*.



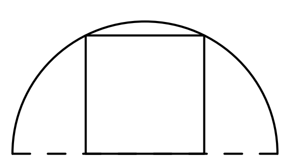
Recall that arc length and sector area for a central angle θ are given by the following formulas:

$$\text{Arc Length} = \theta r$$
$$\text{Sector Area} = \frac{\theta r^2}{2}$$

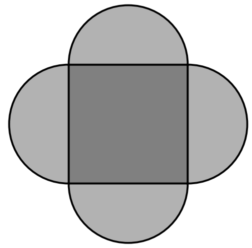
Note that these are simply $\frac{\theta}{2\pi}$ multiplied by the circumference and area, respectively.

1.1 Exercises

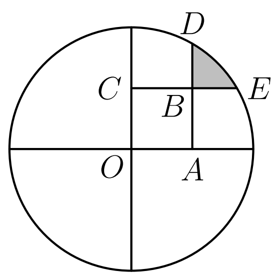
1. **(2006 AMC 10B #8)** A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



2. **(2006 AMC 10B #6)** A region is bounded by semicircular arcs constructed on the side of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region?



3. **(2006 AMC 10B #19)** A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?

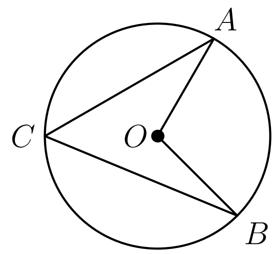


4. **(1997 AIME #4)** Circles of radii $5, 5, 8$, and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.

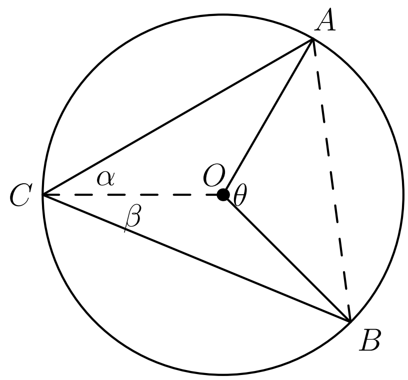
2 Inscribed Angle Theorem

The inscribed angle theorem states that the measure of an inscribed angle is half of the measure of the arc that it intercepts. For example, in the diagram below, we would have

$$m\angle AOB = m\overset{\frown}{AB} = 2 \cdot m\angle ACB.$$

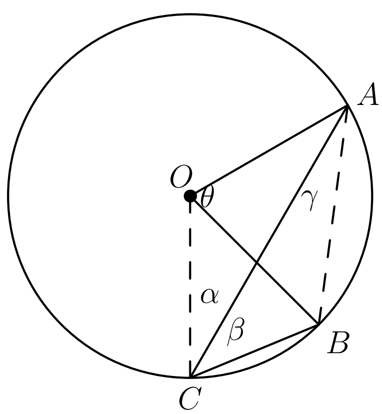


There are two cases we have to show: when the inscribed angle intersects the corresponding central angle, and when it does not. We begin with the case where they do not intersect.



As shown above, we draw the segments $\overline{OC}, \overline{AB}$, and label angles for convenience. From there, we just follow the angles. Since $\triangle AOC$ and $\triangle AOB$ both have two radii, they are isosceles. So, $m\angle AOC = 180^\circ - 2\alpha$ and $m\angle BOC = 180^\circ - 2\beta$. This means that $\theta + 360^\circ - 2\alpha - 2\beta = 360^\circ$. So, $m\angle ACB = \alpha + \beta = \theta/2$.

For the case where they do intersect, we once again draw the lines $\overline{OC}, \overline{AB}$, and label angles:



Then, we have $\triangle AOC, \triangle AOB, \triangle BOC$ are all isosceles, so $m\angle OAC = \alpha$, $m\angle ABO = \alpha + \gamma$, and $m\angle CBO = \alpha + \beta$. Since the internal angles of a triangle always sum to 180°, we have:

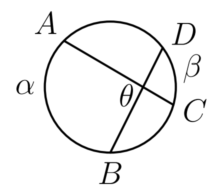
$$\theta + 2\gamma + 2\alpha = m\angle AOB + m\angle OBA + m\angle BAO = 180^\circ = m\angle ABC + m\angle BCA + m\angle CAB = 2\gamma + 2\beta + 2\alpha.$$

Which means that $m\angle CAB = \beta = \theta/2$.

While the statement of the inscribed angle theorem is quite simple, figuring out how to use it can be a bit more tricky. In general, you will often have to create the inscribed angle yourself by drawing some auxiliary lines. Try to create new triangles with the auxiliary lines you draw, and make sure that any inscribed angle that you create intercepts an important arc, especially one that you know the measure of. An important special case to remember is that a diameter/semicircle is always intercepted by an inscribed right angle.

2.1 Examples

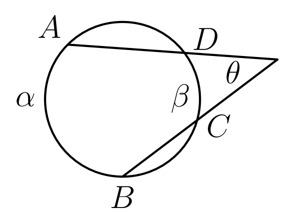
1. **(Intersecting Chords Theorem: Case 1)** Find the angle θ in terms of α and β.



2. **(2020 AMC 12B # 10)** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

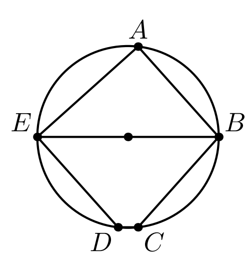
2.2 Exercises

1. **(Intersecting Chords Theorem: Case 2)** Find the angle θ in terms of α and β.



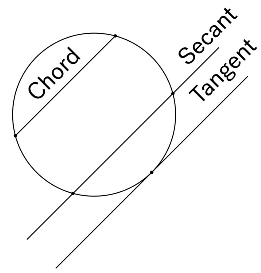
2. **(2014 AMC 12A #12)** Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

3. **(2011 AMC 10B #17)** In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



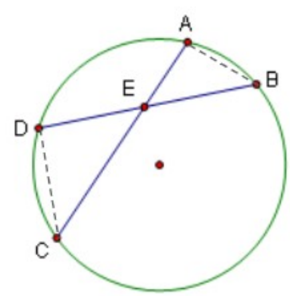
3 Power of a Point

First, recall the three main types of lines/line segments we will deal with when working with circles. A *chord* is a segment connecting two points on a circle. A *secant* is a line that intersects a circle in two points. Another way to think of a secant is as the extension of a chord to a line. A *tangent* is a line that touches (intersects) a circle in exactly one point.



The **power of a point** theorem tells us about the lengths of these when they intersect at the same point. There are three main cases, all of which can be proved using the inscribed angle theorem, and similar triangles:

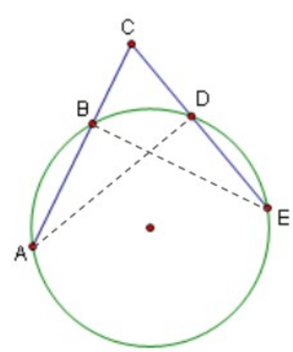
1. Two Chords:



As shown in the diagram above, a good place to start is by drawing the auxiliary lines $\overline{AB}, \overline{CD}$. By the inscribed angle theorem, we have $\angle EAB \cong \angle CDE$ and $\angle ABE \cong \angle DCE$. So, triangles $\triangle EDC$ and $\triangle EAB$ are similar. Thus, we have

$$\frac{AE}{BE} = \frac{ED}{EC} \implies \boxed{AE \cdot EC = BE \cdot ED}.$$

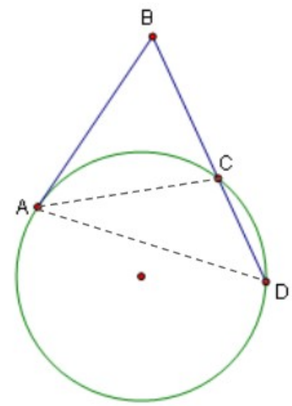
2. Two Secants:



Start by drawing the auxiliary lines $\overline{AD}, \overline{EB}$. By the inscribed angle theorem, we have $\angle BED \cong \angle DAB$. And, since $\angle C$ is shared, $\triangle ACD \sim \triangle ECB$, so:

$$\frac{CB}{CE} = \frac{CD}{CA} \implies \boxed{CB \cdot CA = CD \cdot CE}.$$

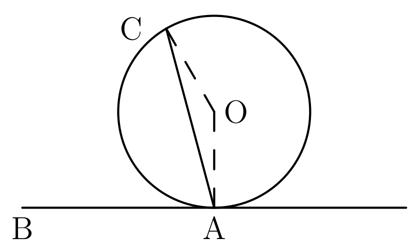
3. Tangent and Secant:



Note that we can treat the point A as a "degenerate" chord, by considering it to be 2 points that coincide. Then, we can apply the intersecting chords theorem to find that $m\angle BAC = \frac{1}{2}m\widehat{AC}$.(*) And, by the inscribed angle theorem, we have that $m\angle BDA = \frac{1}{2}m\widehat{AC}$. Since $\angle B$ is shared, $\angle BAC \sim \angle BDA$, so:

$$\frac{BA}{BC} = \frac{BD}{BA} \implies \boxed{BA^2 = BC \cdot BD}.$$

(*) Exercise: show this without treating A as a "degenerate chord". Consider the figure below, with some helpful auxiliary radii drawn in (the center of the circle has been labeled as "O"):



In general, using power-of-a-point is fairly straightforward. The lines you need to work with will often already be given to you, since some lengths must be known, and any lengths that you are trying to find will be specified by the lines and points that they correspond to.

3.1 Examples

1. **(ARML)** In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

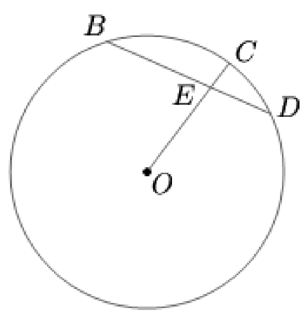
2. Two tangents from an external point P are drawn to a circle and intersect it at A and B. A third tangent meets the circle at T, and the tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at points Q and R, respectively (this means that T is on the minor arc AB). If $AP = 20$, find the perimeter of $\triangle PQR$.

3.2 Exercises

1. **(ARML)** Chords AB and CD of a given circle are perpendicular to each other and intersect at a right angle at point E. Given that $BE = 16$, $DE = 4$, and $AD = 5$, find CE.

2. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.

3. **(1971 CMO #1)** DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given $EC = 1$, find the radius of the circle.

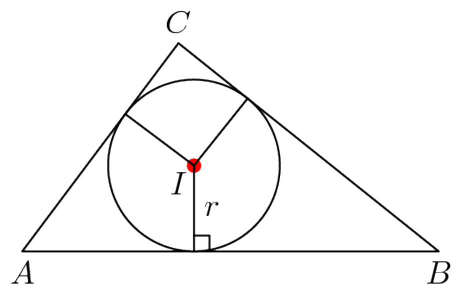


4. **(2005 AIME I #15)** Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

4 Triangle Incircles and Circumcircles

The *incircle* of a triangle is the circle that sits inside the triangle and is tangent to all three sides. Its center is called the *incenter*, and is the point of intersection of the angle bisectors of the triangle. Its radius is called the *inradius* of the triangle, and is usually denoted by a lowercase r.

Consider the following figure:



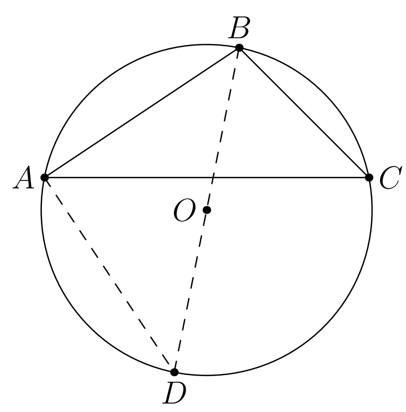
Since each side is tangent to the incircle, the inradius is perpendicular to each of the sides. So, each of the triangles $\triangle AIB, \triangle AIC, \triangle BIC$ has a base equal to one of the sides, and a height equal to the inradius. The sum of their areas is the total area, K. So, we have

$$K = \frac{1}{2}r \cdot AB + \frac{1}{2}r \cdot AC + \frac{1}{2}r \cdot BC = s \cdot r \implies \boxed{r = \frac{K}{s}}.$$

The *circumcircle* of a triangle is the circle which passes through all three vertices. Its center and radius are called the *circumcenter* and *circumradius* (R), respectively. The circumcenter is the point of intersection of the perpendicular bisectors of the triangle, and it may lie in, on, or outside the triangle

- acute triangle $\iff$ circumcenter is inside triangle
- obtuse triangle $\iff$ circumcenter is outside triangle
- right triangle $\iff$ circumcenter is the midpoint of the hypotenuse

Consider $\triangle ABC$ in the diagram below:



We draw the diameter $\overline{BD}$ through the circumcenter O, and we also draw the auxiliary line $\overline{AD}$ to create inscribed angle $\angle D$. Since $\angle C$ and $\angle D$ both intercept arc $\overset{\frown}{AB}$, we have $\angle C \cong \angle D$ by the inscribed angle theorem. Also note that $\angle DAB$ is a right angle, so $c = AB = BD\sin(D) = 2R\sin(C)$.

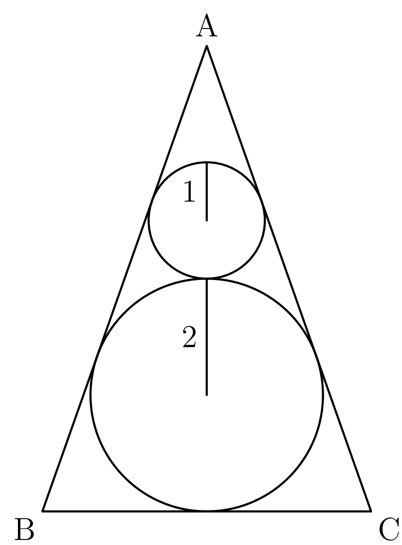
This gives us a very familiar form: $2R = \frac{c}{\sin(C)}$. If we recall the (extended) law of sines:

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)} = \frac{a}{\sin(A)} = \frac{abc}{2K} \implies \boxed{R = \frac{abc}{4K}}.$$

4.1 Examples

1. **(2004 AMC 10B #22)** A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

2. **(2006 AMC 10A #16)** A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



4.2 Exercises

1. **(2010 AIME I #15)** In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Then $\frac{AM}{CM} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

2. **(2011 AIME II #13)** Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a + b$.